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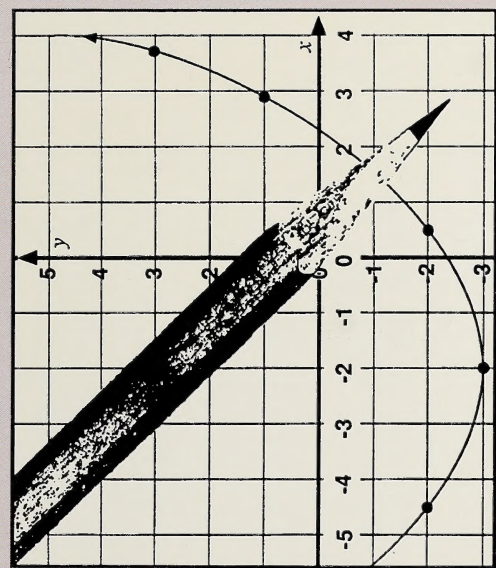


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MATHEMATICS 30

Quadratic Functions
and Equations

Unit 4



Alberta
EDUCATION



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W e l c o m e



Distance Learning

You have chosen an alternate form of learning that allows you to work at your own pace. You will be responsible for your own schedule, for disciplining yourself to study the units thoroughly, and for completing your units regularly. We wish you much success and enjoyment in your studies.

Mathematics 33 Student Module Unit 4 Quadratic Functions and Equations Alberta Distance Learning Centre ISBN No. 0-7741-0224-1

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General Information

This information explains the basic layout of each booklet.

- **What You Already Know and Review** are to help you look back at what you have previously studied. The questions are to jog your memory and to prepare you for the learning that is going to happen in this unit.
- As you begin each **Topic**, spend a little time looking over the components. Doing this will give you a preview of what will be covered in the topic and will set your mind in the direction of learning.
- **Exploring the Topic** includes the objectives, concept development, and activities for each objective. Use your own papers to arrive at the answers in the activities.
- **Extra Help** reviews the topic. If you had any difficulty with **Exploring the Topic**, you may find this part helpful.

- **Extensions** gives you the opportunity to take the topic one step further.
- To summarize what you have learned, and to find instructions on doing the unit assignment, turn to the **Unit Summary** at the end of the unit.

- The **Appendices** include the solutions to **Activities (Appendix A)** and any other charts, tables, etc. which may be referred to in the topics (**Appendix B**, etc.).

Visual Cues

Visual cues are pictures that are used to identify important areas of the material. They are found throughout the booklet.

An explanation of what they mean is written beside each visual cue.



Audiotape

- learning by listening to an audiotape



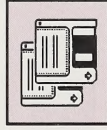
What You Already Know

- reviewing what you already know



Key Idea

- flagging important ideas



Computer Software

- learning by using computer software



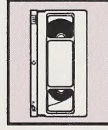
Review

- studying previous concepts



Solutions

- correcting the activities



Videotape

- learning by viewing a videotape



Introduction

- introducing the unit



Extra Help

- providing additional study



Print Pathway

- choosing a print alternative



What Lies Ahead

- previewing the unit



Extensions

- going on with the topic



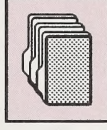
Calculator

- using your calculator



Exploring the Topic

- actively learning new concepts



What You Have Learned

- summarizing what you have learned

Mathematics 33

Course Overview

Mathematics 33 contains 8 units. Beside each unit is a percentage that indicates what the unit is worth in relation to the rest of the course. The units and their percentages are listed below. You will be studying the unit that is shaded.

Unit 1 Powers and Radicals	10%
Unit 2 Polynomials and Rational Expressions	10%
Unit 3 Functions and Relations	16%
Unit 4 Quadratic Functions and Equations	20%
Unit 5 Trigonometry	16%
Unit 6 Statistics	16%
Unit 7 Annuities	6%
Unit 8 Mortgages and Loans	6%
	100%

Unit Assessment

After completing the unit you will be given a mark based totally on a unit assignment. This assignment will be found in the Assignment Booklet.

Unit Assignment - 100%

If you are working on a CML terminal, your teacher will determine what this assessment will be. It may be

Unit Assignment - 50%
Supervised Unit Test - 50%

Introduction to Quadratic Functions and Equations

This unit covers topics dealing with quadratic functions and equations. Each topic contains explanations, examples, and activities to assist you in understanding quadratic functions and equations. If you find you are having difficulty with the explanations and the way the material is presented, there is a section called **Extra Help**. If you would like to extend your knowledge of the topic, there is a section called **Extensions**.

You can evaluate your understanding of each topic by working through the activities. Answers are found in the solutions in **Appendix A**. In several cases there is more than one way to do the question.

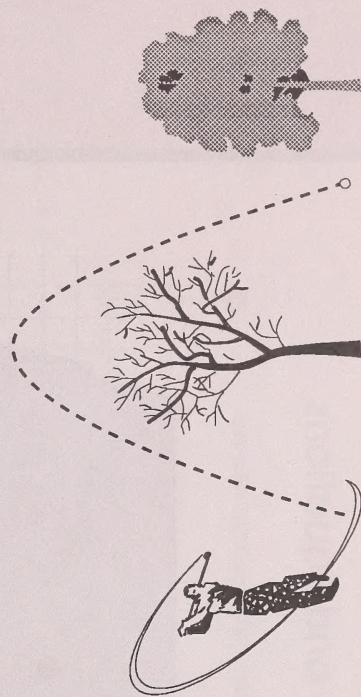
Unit 4 Quadratic Functions and Equations

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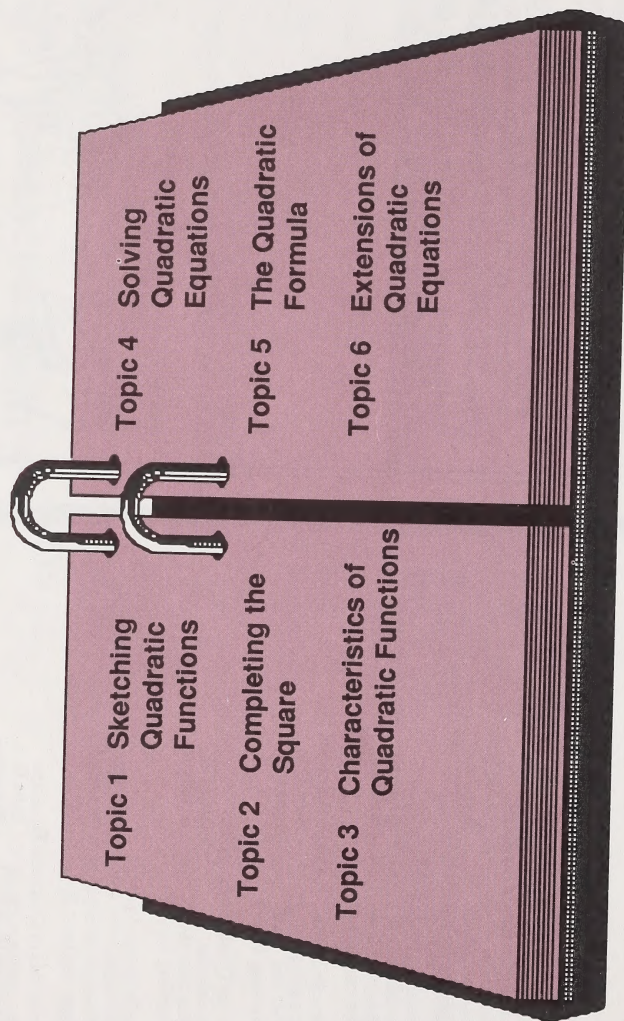
Quadratic Functions and Equations

Parabolas are a common shape in the world around us. They can be described mathematically in simple terms, yet they provide an invitation for some fascinating mathematical explorations. In this unit you will study parabolas and the equations that describe them.



Unit 4

Quadratic Functions and Equations



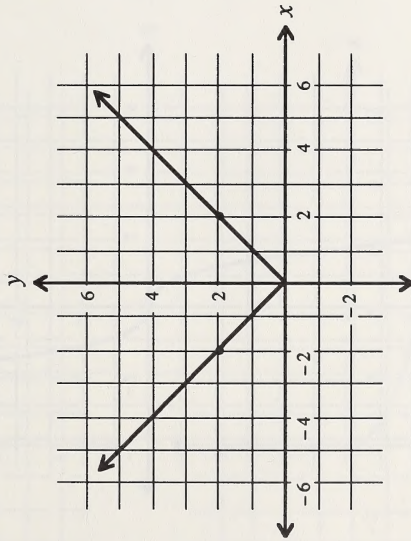


What You Already Know

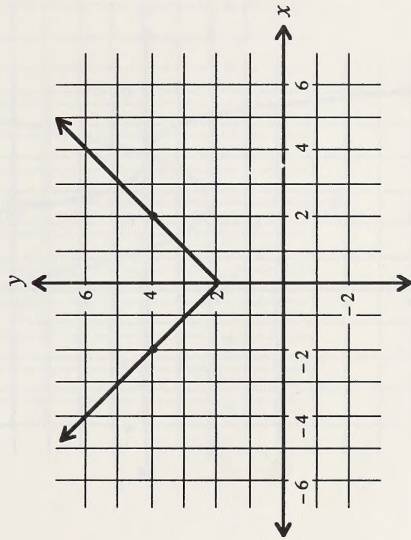
Refresh your memory!

- The graph of $y = f(x) + b$ is a vertical translation of the graph of $y = f(x)$ by b units.

$$y = |x|$$



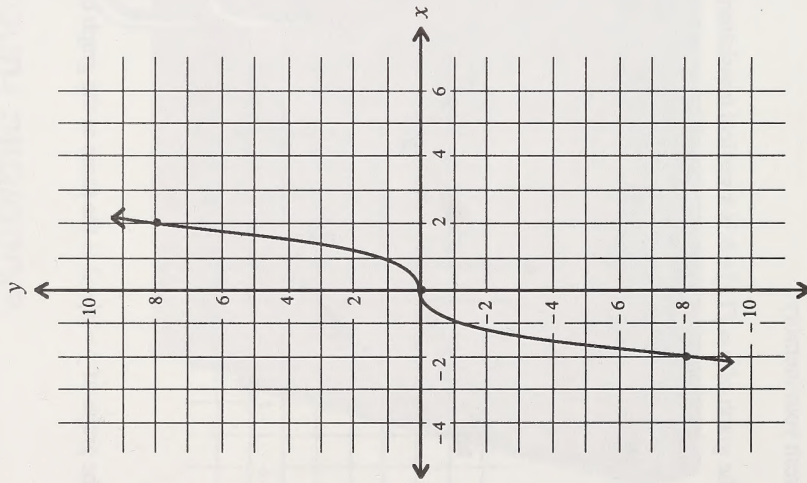
$$y = |x| + 2$$



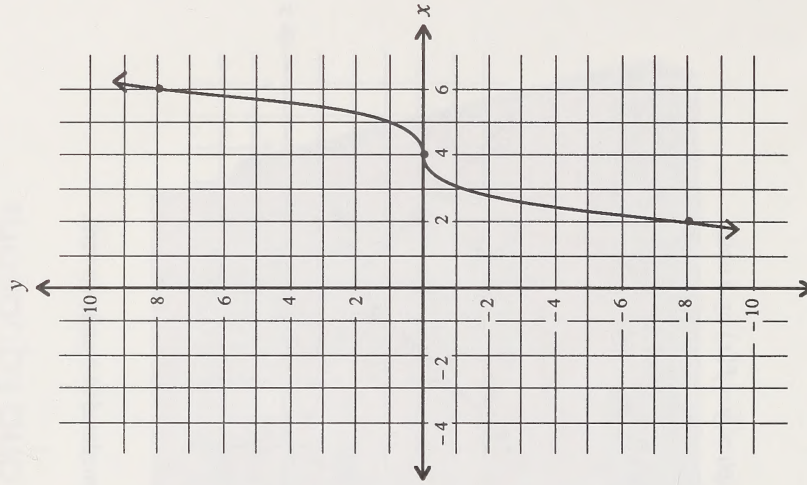
The graph of $y = |x| + 2$ is the same as the graph of $y = |x|$ translated two units upward.

- The graph of $y = f(x - a)$ is a horizontal translation of the graph of $y = f(x)$ by a units.

$$y = x^3$$



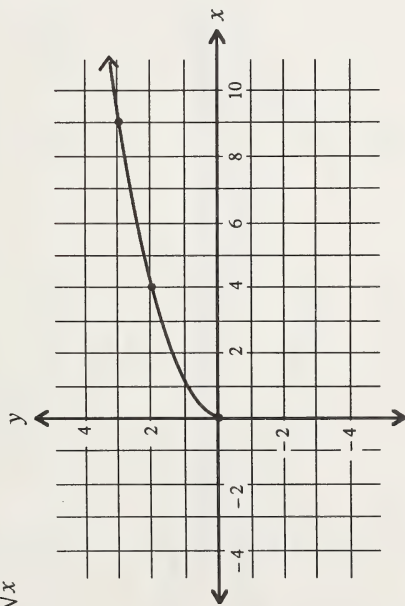
$$y = (x - 4)^3$$



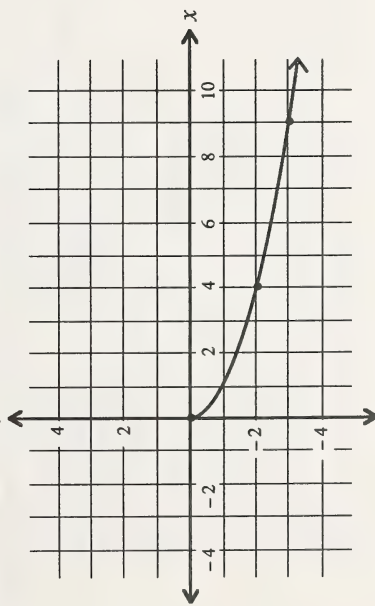
The graph of $y = (x - 4)^3$ is the same as the graph of $y = x^3$ translated four units to the right.

- The graph of $y = -f(x)$ is a reflection in the x -axis of the graph of $y = f(x)$.

$$y = \sqrt{x}$$



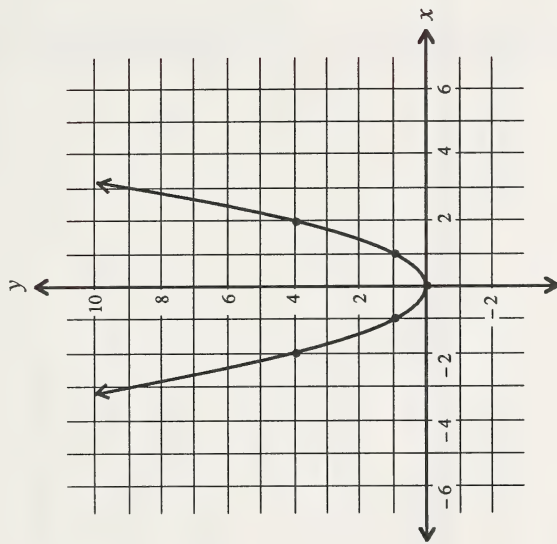
$$y = -\sqrt{x}$$



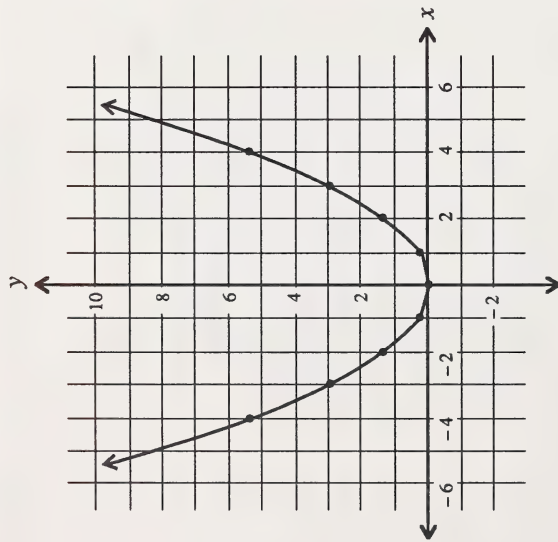
The graph of $y = -\sqrt{x}$ is the same as the graph of $y = \sqrt{x}$ reflected in the x -axis.

- The graph of $y = cf(x)$ is a vertical stretch or compression of the graph of $y = f(x)$.

$$y = x^2$$



$$y = \frac{1}{3}x^2$$



The graph of $y = \frac{1}{3}x^2$ has the same general shape as the graph of $y = x^2$ except that it has been vertically compressed (made wider) by a factor of 3.

- The equation of a horizontal line passing through (h, k) is $y = k$.
The equation of a vertical line passing through (h, k) is $x = h$.
The horizontal line passing through $(2, 3)$ is described by the equation $y = 3$.
The vertical line passing through $(2, 3)$ is described by the equation $x = 2$.

- Perfect-square trinomials can be factored into a binomial that is squared.

$$x^2 + 6x + 9 = (x + 3)^2 \qquad x^2 - 10x + 25 = (x - 5)^2$$

- Most polynomials can be factored. You should be aware of the different factoring procedures.

$$2x^2 + 6x + 10 = 2(x^2 + 3x + 5)$$

$$3x^2 - 6x = 3x(x - 2)$$

$$x^2 - 16 = (x + 4)(x - 4)$$

$$x^2 - 7x + 12 = (x - 3)(x - 4)$$

$$2x^2 + 9x + 4 = (2x + 1)(x + 4)$$

- A radical in simplest form contains the smallest possible radicand.

$$\begin{aligned}\sqrt{50} &= \sqrt{25 \times 2} \\ &= \sqrt{25} \times \sqrt{2} \\ &= 5 \times \sqrt{2} \\ &= 5\sqrt{2}\end{aligned}$$

- Equations with rational expressions may contain nonpermissible values.

$$\begin{array}{lcl}\frac{2x+3}{3x-7} = \frac{2x-5}{x+3} & 3x-7 \neq 0 & x+3 \neq 0 \\ 3x \neq 7 & 3x \neq 7 & x \neq -3 \\ x \neq \frac{7}{3} & & \end{array}$$

The nonpermissible values are $\frac{7}{3}$ and -3 ($x \neq \frac{7}{3}, -3$).

Now that you have looked at material you studied previously, go to the **Review** to confirm your understanding of this material.



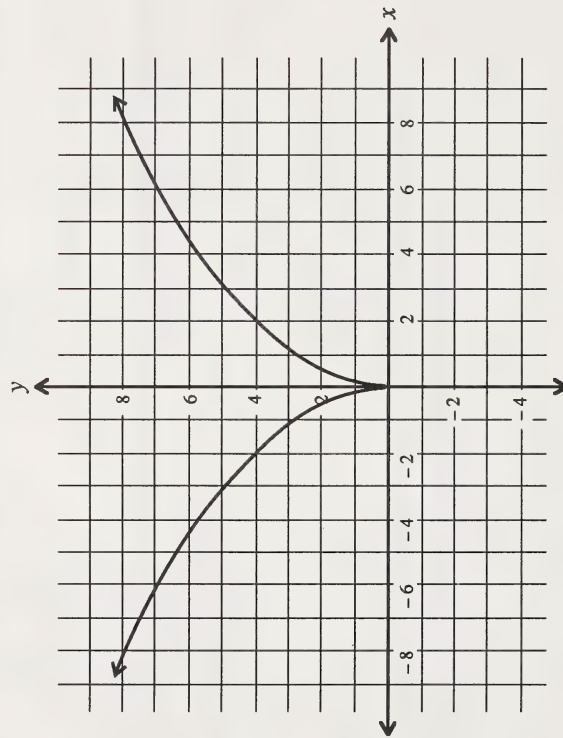
Review

Try the following questions. Graph paper is provided in Appendix B.

1. The graph of $y = f(x)$ is shown. Sketch the graph of the following:

a. $y = f(x) + 4$

b. $y = f(x) - 3$



2. Using the graph in question 1, sketch the graph of the following:

a. $y = f(x - 4)$

b. $y = f(x + 3)$

3. Using the graph in question 1, sketch the graph of $y = -f(x)$.

4. Using the graph in question 1, sketch the graph of the following:

a. $y = 2f(x)$

b. $y = \frac{1}{2}f(x)$

5. Write the equation of the horizontal line and the vertical line passing through the following points.

a. $(5, 3)$

b. $(-2, 5)$

6. Factor each perfect square trinomial.

a. $x^2 - 6x + 9$

b. $x^2 + 2x + 1$

c. $x^2 + \frac{4}{3}x + \frac{4}{9}$

d. $x^2 - \frac{4}{3}x + \frac{4}{9}$

7. Factor each polynomial.

a. $5x^2 - 10y$

b. $2x^2 - 10x$

c. $x^2 - 25$

d. $4x^2 - 9$

e. $x^2 + 2x - 15$

f. $x^2 - 5x - 14$

g. $3x^2 - 8x + 4$

h. $2x^2 - x - 6$

8. Simplify.

a. $\sqrt{18}$

b. $2\sqrt{32}$

c. $\sqrt{8} + \sqrt{12}$

d. $3\sqrt{27}$

9. Find the nonpermissible values for each rational expression.

a. $\frac{3x-7}{2x+5}$

b. $\frac{3x-7}{x-4}$

c. $\frac{3x-7}{2x}$



Now go to the **Review solutions in Appendix A.**

If you had difficulty with questions 1 to 4, refer to **Mathematics 33, Unit 3, Functions and Relations.**

If you had difficulty with question 5, refer to **Mathematics 23, Unit 4, Linear Relations.**

If you had difficulty with questions 6 and 7, refer to **Mathematics 10, Unit 4, Factoring Polynomials.**

If you had difficulty with question 8, refer to **Mathematics 33, Unit 1, Powers and Radicals.**

If you had difficulty with question 9, refer to **Mathematics 33, Unit 2, Polynomials and Rational Expressions.**

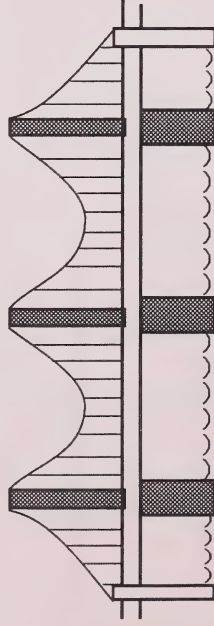
Topic 1 Sketching Quadratic Functions



Introduction

In Unit 3 you learned how to transform and sketch a variety of relations and functions.

In this topic you will learn how to transform and sketch a special type of function called the **quadratic function**. The graph of a quadratic function is a **parabola**. Can you find a parabola in the following diagram?



What Lies Ahead

Throughout the topic you will learn to

1. sketch the graphs of quadratic functions written in the form $y = a(x - h)^2 + k$

Now that you know what to expect, turn the page to begin your study of sketching quadratic functions.



Exploring Topic 1

Throughout this unit you may use the computer program *Graphing Equations*¹ to draw the graphs instead of sketching by hand. Follow the instructions on the computer program.



Activity 1



Sketch the graphs of quadratic functions written in the form $y = a(x - h)^2 + k$.

It is possible to draw the graph of any function or relation provided the equation is given. The following example shows how this is done.

Example 1

Draw the graph of the function $y = x^2$.

¹ *Graphing Equations* is a title of Conduit.

Solution:

Give values to x (the independent variable) and solve for the corresponding y -value (dependent variable).

$$\text{When } x = 0, y = 0^2$$

$$= 0$$

$$\text{When } x = 1, y = 1^2$$

$$= 1$$

$$\text{When } x = -1, y = (-1)^2$$

$$= 1$$

$$\text{When } x = 2, y = (2)^2$$

$$= 4$$

$$\text{When } x = -2, y = (-2)^2$$

$$= 4$$

$$\text{When } x = 3, y = (3)^2$$

$$= 9$$

$$\text{When } x = -3, y = (-3)^2$$

$$= 9$$

This information may be put in a table as shown.

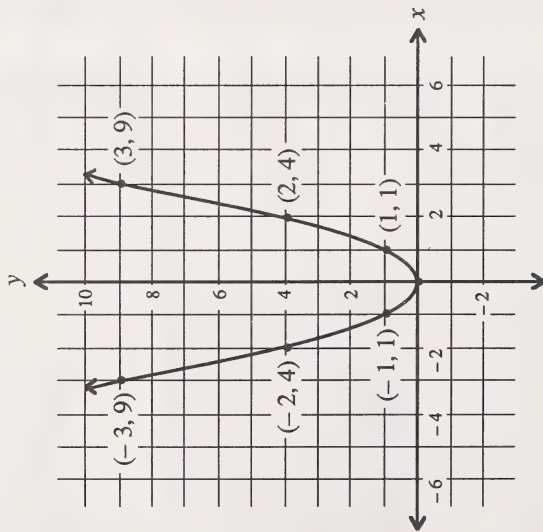
x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

When giving values to x , use negative values as well as positive values.

Note: The independent variable, usually x , is put on the top line of the table and the dependent variable, usually y , is put on the bottom.

Another way to show this information is by using ordered pairs. The information in the table can be shown as $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(2, 4)$, $(-2, 4)$, $(3, 9)$, $(-3, 9)$.

A third way to show this information is by plotting the ordered pairs on a coordinate plane and joining the points to form a curve. The following graph shows what the curve would look like.

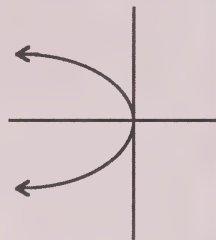


This curve is called a parabola.



The ordered pair $(3, 9)$ means when $x = 3$, $y = 9$.

The arrows mean that the graph continues in the upward direction.



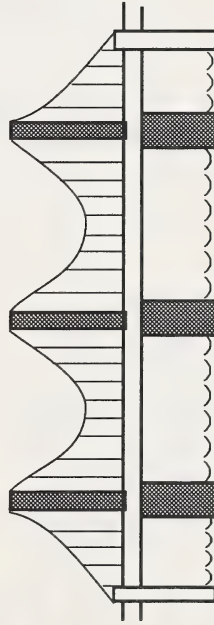
You are probably aware of some curves that take the shape of parabolas. The following are some examples.



a flare projected upward



a TV satellite dish



the cables in a suspension bridge



a vehicle's headlights

Parabolas can open downward (as with the flare), upward (as with the suspension bridge), or sideways (as with the headlights). In this topic you will only be dealing with parabolas opening upward or downward.

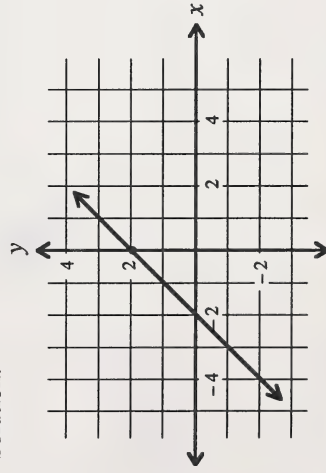
Is it possible to predict whether the graph of a certain kind of function will be a parabola? Is there a characteristic common to all functions whose graphs are parabolas? To answer these questions, you should graph some more functions to see if you can identify a common characteristic. Follow the next examples.

Example 2

Which of the following functions will form a parabola when graphed?

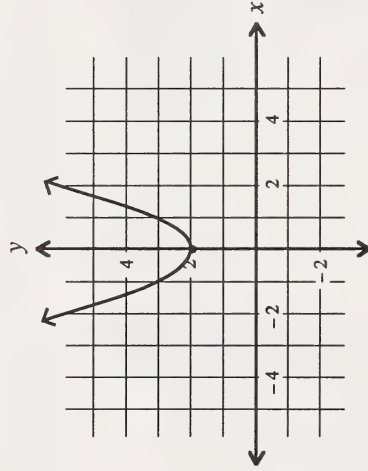
- $y = x + 2$

Solution:



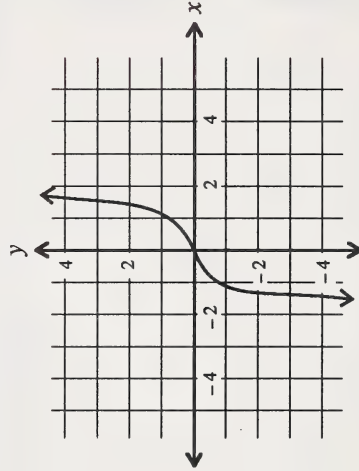
- $y = x^2 + 2$

Solution:



- $y = x^3$

Solution:

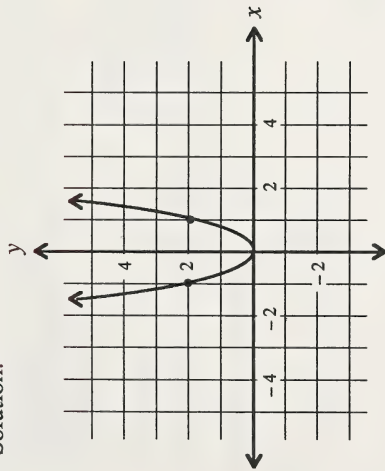


Note: Parabolas that open upward or downward are functions. They pass the vertical-line test. Parabolas that open sideways are not functions.

You may sketch these graphs by completing a table of values or by using a computer or a graphing calculator.

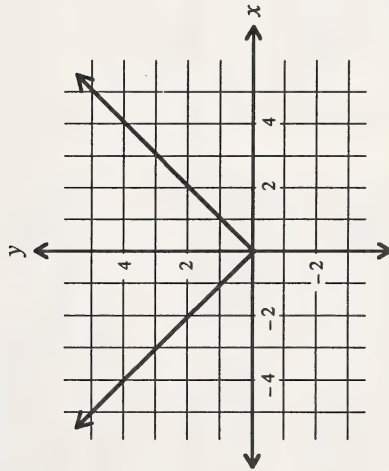
- $y = 2x^2$

Solution:



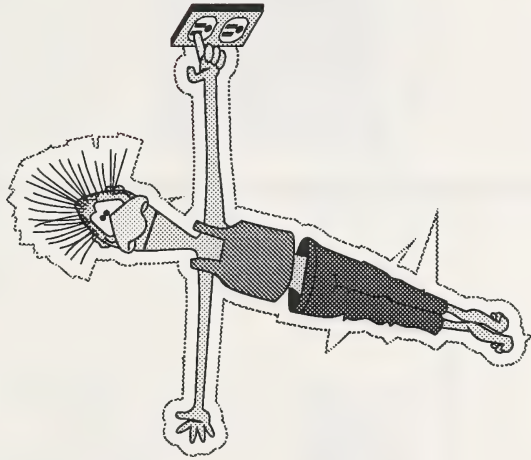
- $y = |x|$

Solution:



The functions $y = x^2 + 2$ and $y = 2x^2$ result in parabolas when graphed.

You can see that the functions $y = x^2$, $y = x^2 + 2$, and $y = 2x^2$ will form parabolas when graphed. The characteristic that is common to all three functions is that each contains an x^2 term.





Functions like $y = x^2$, $y = x^2 + 2$, and $y = 2x^2$ belong to a group of functions called **quadratic functions**.



A **parabola** is the shape that is characteristic of the graphs of all quadratic functions.

Note that each of the functions $y = x^2$, $y = x^2 + 2$, and $y = 2x^2$ is of **degree two**. All quadratic functions are of **degree two**. Since quadratic functions are of degree two, they must contain a variable that is squared (x^2 , s^2 , or t^2 , for example). Some examples of quadratic functions are as follows:

$$y = 3x^2 + 4, h = -3t^2 + 16, y = x^2 + 5x - 6,$$

$$m = 3n^2 + 2n$$

The following two facts are important.

- All quadratic functions are of degree two.
- A parabola is the characteristic shape of the graphs of all quadratic functions.

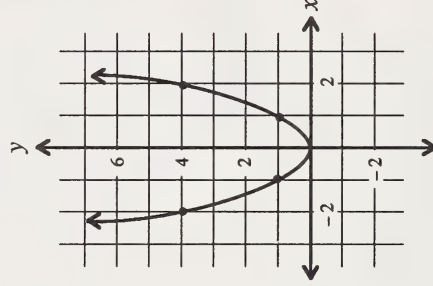


To begin a more detailed study of quadratic functions, you will begin by looking at the reference function and the reference parabola.

The **reference quadratic function** is as follows:

$$y = x^2$$

The **reference parabola** is as follows:



Refer to Example 1 of this topic to see how the previous parabola was obtained from the function $y = x^2$.

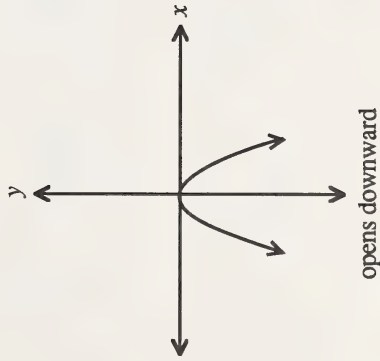
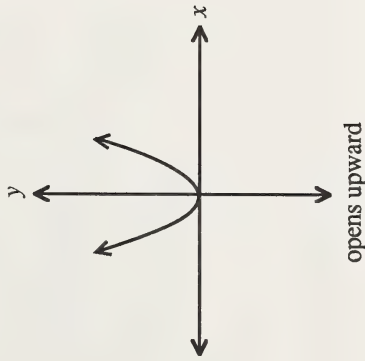
Variables other than x and y may be used. For example, $k = m^2$, $z = w^2 + 2$, and $p = 2r^2$ are all quadratic functions.

The parabola obtained from $y = x^2$ is called the **reference parabola** because all other parabolas can be obtained from it. You should make yourself familiar with this shape.

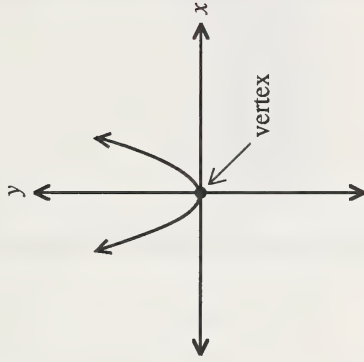
In your study of parabolas you will need to be aware of the following facts.



Some parabolas open upward and some open downward.



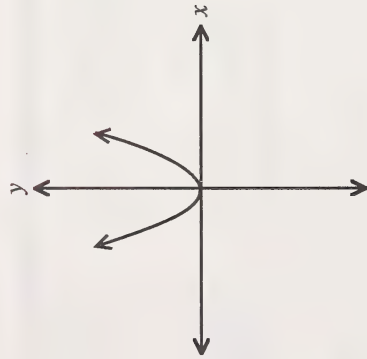
The **vertex** of a parabola is the point at which the parabola changes direction. This point can be described by an **ordered pair**.



The vertex for this parabola has the ordered pair $(0, 0)$.



The **axis of symmetry** is a **vertical line** that divides the parabola into two parts which are mirror reflections of each other. The axis of symmetry must run through the vertex.



To answer the previous questions, you can sketch the graph of $y = x^2 + 3$ and then compare it to the graph of $y = x^2$.

There are two ways that you can sketch the graph of $y = x^2 + 3$.

Method 1: Make a table of values and plot the ordered pairs. (See Example 2 for the method.)

$$y = x^2 + 3$$

x	y
-3	12
-2	7
-1	4
0	3
1	4
2	7
3	12

The vertex is at $(0, 0)$. Therefore, the y -axis is the axis of symmetry. The equation that describes the y -axis is $x = 0$.

The reference parabola has the following characteristics.

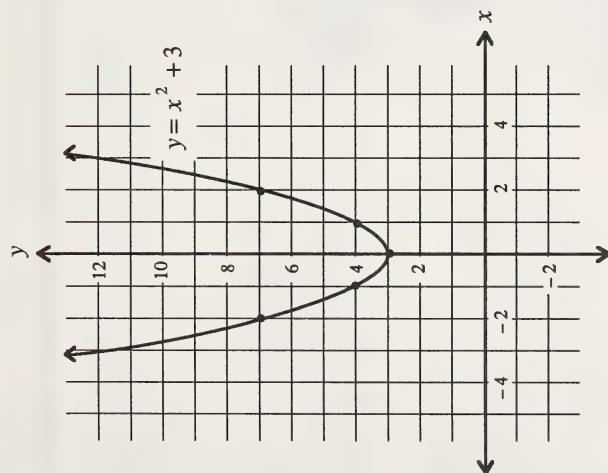
- It opens upward.
- The vertex is at $(0, 0)$.
- The axis of symmetry is the vertical line defined by $x = 0$ (the y -axis).

If a change is made to the reference function, how will the reference parabola be affected?

For example, how do the graphs of $y = x^2 + 3$ and $y = x^2$ compare? Are there any similarities? Are there any differences?

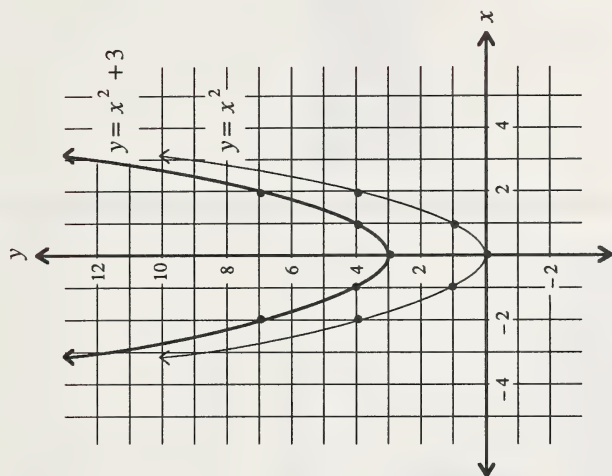
You may also use a computer or a graphing calculator to generate the graph of $y = x^2 + 3$.

The point $(0, 0)$ is commonly referred to as the **origin**.



Method 2: Recall from the previous unit that adding a constant to a function just translates the graph of that function vertically upward or downward.

The graph of $y = x^2 + 3$ is the same as the graph of $y = x^2$ translated three units upward.



You can see that both methods result in the same graph. Since you learned how to translate graphs in the last unit, you will continue to use translations in this unit (Method 2).

When comparing the graphs of $y = x^2 + 3$ and $y = x^2$, you see that the graph of $y = x^2 + 3$ is the same as the graph of $y = x^2$ except that it has been translated three units upward. Both graphs open upward, and they have the same shape, and the same axis of symmetry, but they have different vertices. The vertex for $y = x^2$ is at

$(0, 0)$, while the vertex for $y = x^2 + 3$ is at $(0, 3)$.

Using your knowledge of translations, could you predict what the graph of $y = x^2 - 2$ would look like?

If you said that the graph of $y = x^2 - 2$ is the same as the graph of $y = x^2$ translated two units downward, you would be correct. The vertex of the new graph would be at $(0, -2)$, but otherwise both graphs would be the same.



The graph of $y = x^2 + k$ is the same as the graph of $y = x^2$, except that it has been translated k units vertically.

If k is positive, the following is true:

- The reference parabola is translated k units up.
- The vertex is at $(0, k)$.

If k is negative, the following is true:

- The reference parabola is translated k units down.
- The vertex is at $(0, k)$.

Follow the next example to see how you may quickly sketch the parabola of a quadratic function in the form $y = x^2 + k$.



Recall: The reference parabola is the graph of $y = x^2$.

Vertices is the plural of vertex.

Note: The variable k is any real number.

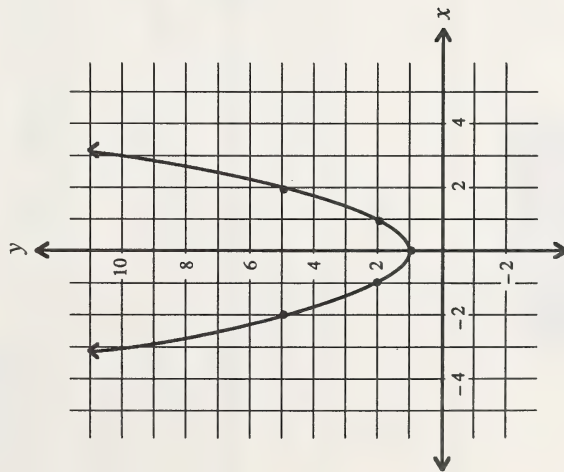
Example 3

Using your knowledge of translations, sketch the graphs of the following.

- $y = x^2 + 1$

Solution:

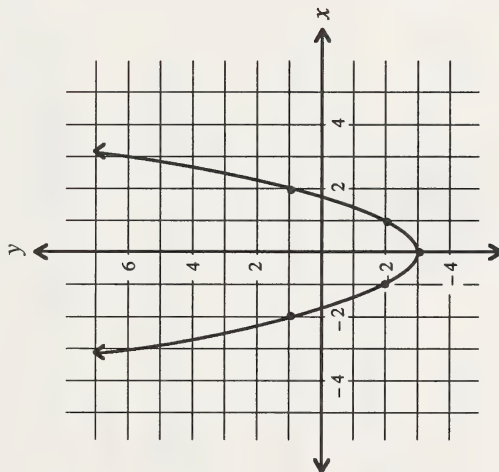
The graph will be the same as the graph of $y = x^2$, except it will be translated upward one unit.



- $y = x^2 - 3$

Solution:

The graph will be the same as the graph of $y = x^2$, except it will be translated downward three units.



When giving values to x , use negative values as well as positive values.

You should look at some of the quadratic functions that you have encountered so far. Can you tell what the coordinates of the vertex are by looking at the equation of a function?

Recall: The equation $x = 0$ represents the y -axis.

Direction of opening Axis of symmetry Vertex coordinates	$y = x^2$	$y = x^2 + 1$	$y = x^2 - 3$
	upward $x = 0$ (0, 0)	upward $x = 0$ (0, 1)	upward $x = 0$ (0, -3)

The second coordinate of the vertex has the same value as the constant in the function.

- $y = x^2$ The vertex is at (0, 0).
- $y = x^2 + 1$ The vertex is at (0, 1).
- $y = x^2 - 3$ The vertex is at (0, -3).

You can make predictions about the graphs of quadratic functions by just looking at them.

- The function $y = x^2 + 7$ is the reference parabola translated seven units upward. The vertex is at (0, 7).
- The function $y = x^2 - 5$ is the reference parabola translated five units downward. The vertex is at (0, -5).

Adding or subtracting a constant from $y = x^2$ results in a vertical translation of the reference parabola.

It is time for you to test your knowledge of the concepts studied so far in this topic. Do the following exercises. Graph paper is provided in **Appendix B** for questions that require it. You may use a computer program or a graphing calculator to help you answer the questions that ask you to sketch graphs.

- By completing a table of values and plotting the ordered pairs, draw the graph of the function defined by $y = 2x^2 - 3$.
- What is the degree of a quadratic function?
- What is the characteristic shape of the graph of a quadratic function?
- Write the equation for the reference quadratic function.
- What is the vertex of a parabola?
- What is the axis of symmetry of a parabola?
- For the reference parabola, what are the coordinates of the vertex and what is the equation for the axis of symmetry?
- Describe the difference between the graphs of the following.

- $y = x^2 + 8$ and $y = x^2$
- $y = x^2 - 5$ and $y = x^2$

- Write the coordinates of the vertex for each function that follows.

- $y = x^2 + 1$
- $y = x^2 - 4$

- By translating the graph of $y = x^2$, sketch the graph of any two functions that follow. Identify the coordinates of the vertex for each graph.

- $y = x^2 + 2$
- $y = x^2 - 2$
- $y = x^2 + 5$
- $y = x^2 - 5$

- The graph of $y = x^2 + k$ is a _____ translation of the graph of $y = x^2$ by k units.



For solutions to **Activity 1**, turn to **Appendix A**, **Topic 1**.

The graph of the reference function is translated vertically by adding or subtracting a constant. What changes must be made to the reference function in order for a horizontal translation to occur? Is it possible to identify a horizontal translation by looking at the equation of a function? To answer these questions, look at the following example.

Example 4

Sketch the graph of $y = (x - 3)^2$.

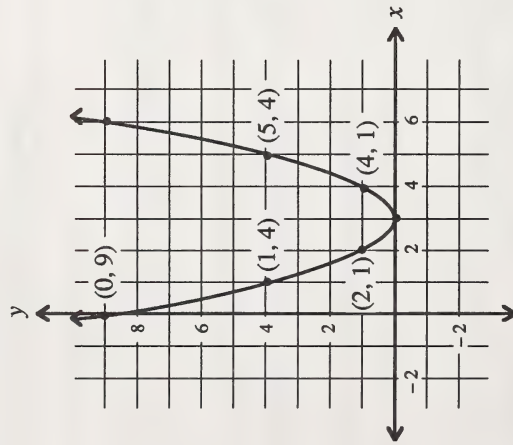
Solution:

Make a table of values and plot the ordered pairs.

$$y = (x - 3)^2$$

x	-1	0	1	2	3	4	5
y	16	9	4	1	0	1	4

The graph is as follows.



This graph is a parabola, and it has the same shape as the reference parabola except that it is translated three units to the right. The vertex of this parabola is at $(3, 0)$.

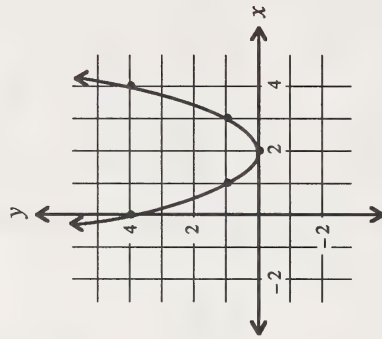
Is it possible to predict what the graph of $y = (x - 3)^2$ will look like by looking at the equation? It appears that the 3 in $y = (x - 3)^2$ causes the reference parabola to be translated three units to the right. Look at the next example to see if this assumption is correct.

Example 5

Sketch the following graphs and draw a conclusion regarding horizontal translations.

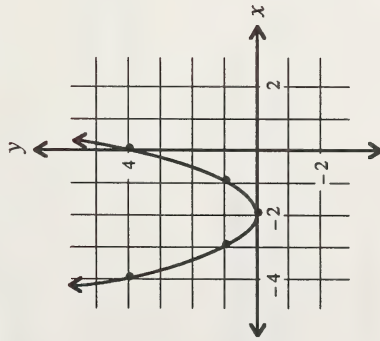
- $y = (x - 2)^2$

Solution:



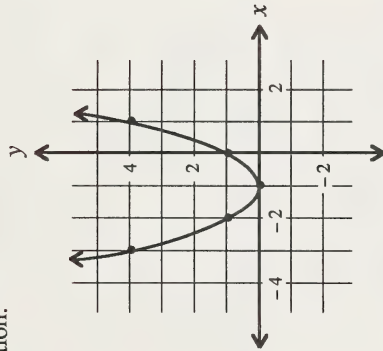
- $y = (x + 2)^2$

Solution:



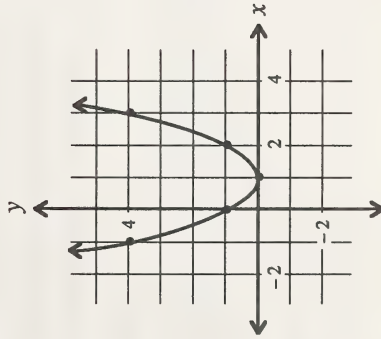
- $y = (x + 1)^2$

Solution:



- $y = (x - 1)^2$

Solution:



It appears that the number inside the brackets causes a horizontal translation of the reference parabola either to the right or to the left. The vertex is also translated horizontally.

The graph of the function

$$y = (x - h)^2$$

is the same as the graph of $y = x^2$ translated h units horizontally to the right if h is positive. If h is negative, the graph is translated h units horizontally to the left. The vertex is at $(h, 0)$.



To verify these graphs, you may complete a table of values or you may use a computer.

Note the relationship between the equation and the vertex.

For $y = (x - 7)^2$, $h = 7$ and the vertex is at $(7, 0)$.

For $y = (x + 5)^2$, $h = -5$ since you can write $y = [x - (-5)]^2$ and the vertex is at $(-5, 0)$.

Also note that the axis of symmetry for

$y = (x - h)^2$ is defined by $x = h$ so for

$y = (x - 7)^2$, the axis of symmetry is $x = 7$ since $h = 7$.

For $y = (x + 5)^2$, the axis of symmetry is $x = -5$ since $h = -5$.

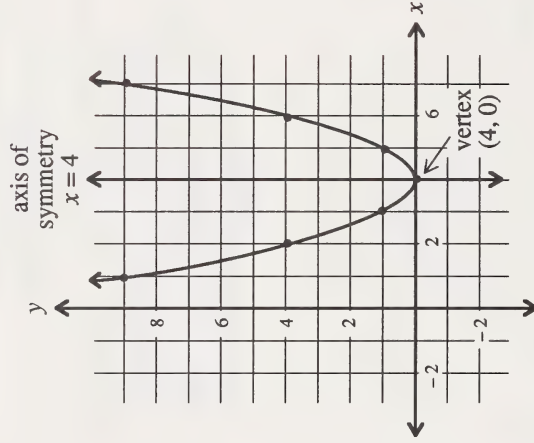
Follow the next example to see how to sketch the graph of a quadratic function using horizontal translation.

Example 6

For the quadratic function $y = (x - 4)^2$, do the following.

- Sketch the graph.

Solution:



A subtraction sign inside the brackets means the parabola is moved to the right. An addition sign inside the brackets means the parabola is moved to the left.

- Explain the relationship between this graph and the graph of $y = x^2$.

Solution:

The graph of $y = (x - 4)^2$ is the same as the graph of $y = x^2$ translated four units to the right.

- Name the coordinates of the vertex.

Solution:

The vertex is at $(4, 0)$.

- Name the equation of the axis of symmetry.

Solution:

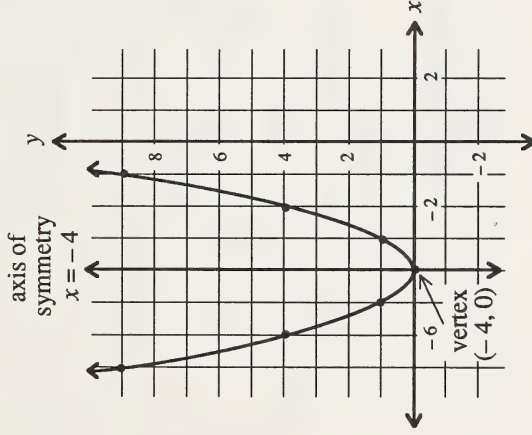
The equation of the axis of symmetry is $x = 4$.

Example 7

For the quadratic function $y = (x + 4)^2$, do the following.

- Sketch the graph.

Solution:



- Explain the relationship between this graph and the graph of $y = x^2$.

Solution:

The graph of $y = (x + 4)^2$ is the same as the graph of $y = x^2$ translated four units to the left.

Note the relationship between the equation of the axis of symmetry and the x-coordinate of the vertex.

- Name the coordinates of the vertex.

Solution:

The vertex is at $(-4, 0)$.

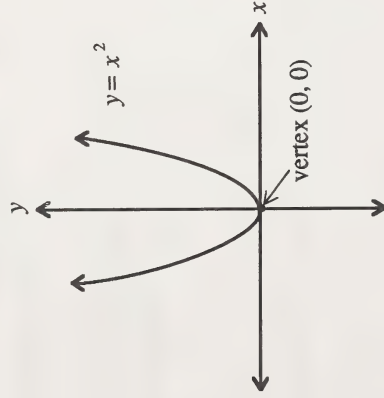
- Name the equation of the axis of symmetry.

Solution:

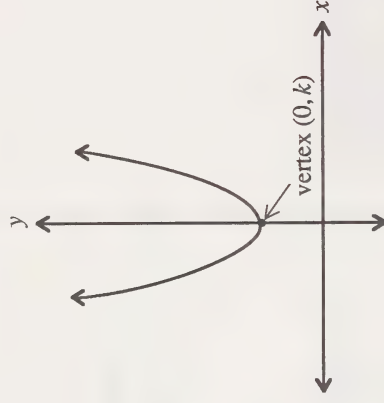
The equation of the axis of symmetry is $x = -4$.

A brief summary of how to sketch the graphs of certain quadratic functions using vertical and horizontal translations is given as follows.

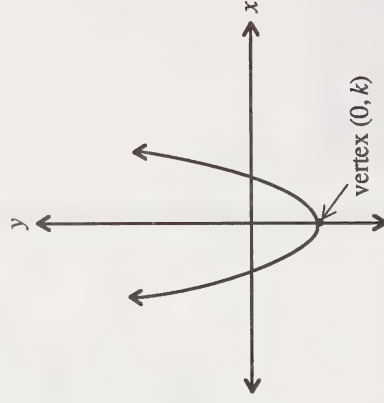
- The graph of the reference function $y = x^2$ is shown.



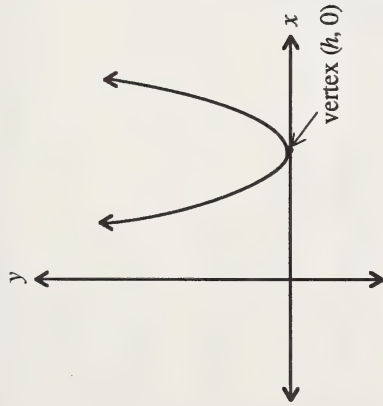
- The graph of $y = x^2 + k$ is the same as the graph of $y = x^2$ translated k units upward if k is positive.



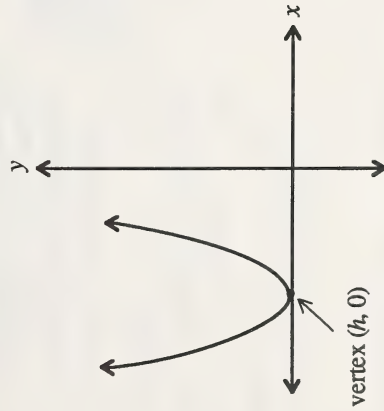
- The graph of $y = x^2 - k$ is the same as the graph of $y = x^2$ translated k units downward if k is negative.



- The graph of $y = (x - h)^2$ is the same as the graph of $y = x^2$ translated h units to the right if h is positive.



- The graph of $y = (x + h)^2$ is the same as the graph of $y = x^2$ translated h units to the left if h is negative.



Do the following exercises on horizontal translations. Graph paper is provided in **Appendix B**.

- The graph of $y = (x - h)^2$ is a _____ translation of the graph of $y = x^2$ by h units.
- Describe the difference between the graphs of the following.
 - $y = (x - 7)^2$ and $y = x^2$
 - $y = (x + 10)^2$ and $y = x^2$
- Write the coordinates of the vertex for each function that follows.
 - $y = (x + 3)^2$
 - $y = (x - 2)^2$
- For $y = (x - h)^2$, the vertex has coordinates of (,).
 - For $y = (x + h)^2$, the vertex has coordinates of (,).
- By translating the graph of $y = x^2$, sketch the graph of any two functions that follow. Identify the coordinates of the vertex for each graph.
 - $y = (x + 6)^2$
 - $y = (x - 5)^2$
 - $y = (x - 1)^2$
 - $y = (x + 2)^2$



For solutions to **Activity 1**, turn to **Appendix A, Topic 1**.

Now you will learn about sketching quadratic functions by translating the reference parabola vertically and horizontally. You will do this as an investigation on your own.

You will investigate the relationship between the quadratic function $y = (x - h)^2 + k$ and the reference function $y = x^2$ by sketching several graphs and drawing a conclusion. Graph paper is provided in **Appendix B**.

You may use the computer program *Graphing Equations*¹ or a graphing calculator to draw the graphs instead of sketching by hand. Follow the instructions on the computer program.



17. Sketch the graphs of the following:

- a. $y = x^2$ b. $y = x^2 + 3$
 c. $y = (x - 2)^2$ d. $y = (x - 2)^2 + 3$

18. After you have sketched the graphs, give a statement explaining how the graph of $y = (x - 2)^2 + 3$ can be obtained from the graph of $y = x^2$.

19. Generalize how the graph of $y = (x - h)^2 + k$ can be obtained from the graph of $y = x^2$.



For solutions to **Activity 1**, turn to **Appendix A, Topic 1**.

From the investigation you generalized that the graph of $y = (x - h)^2 + k$ can be obtained from the graph of $y = x^2$ by a vertical translation of k units and a horizontal translation of h units. Several more graphs have been sketched to see if this generalization holds true. Follow the examples that follow.

Note: You may graph

$y = (x - 2)^2 + 3$ by using the following:

- a table of values
- a computer or a graphing calculator
- the concepts that you have learned in this topic

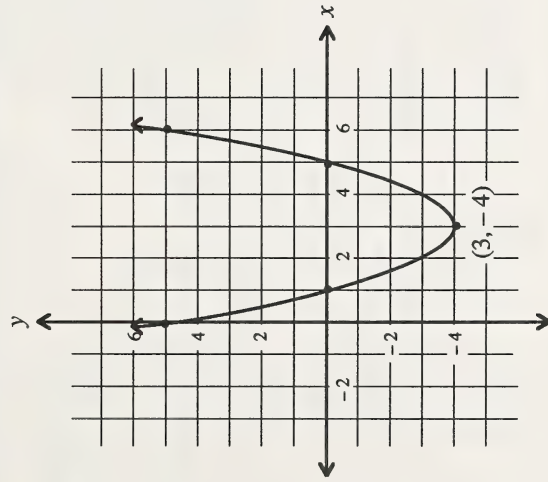
¹ *Graphing Equations* is a title of Conduit.

Example 8

Sketch the graph of each function that follows and identify the coordinates of the vertex.

- $y = (x - 3)^2 - 4$

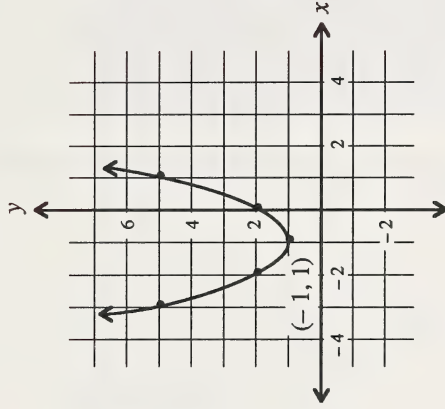
Solution:



The vertex is at $(3, -4)$.

- $y = (x + 1)^2 + 1$

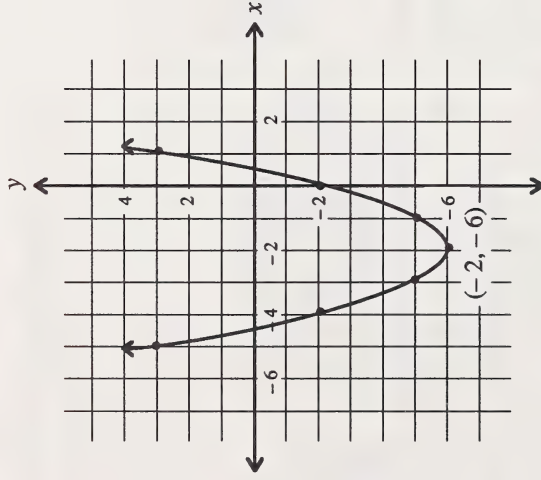
Solution:



The vertex is at $(-1, 1)$.

- $y = (x + 2)^2 - 6$

Solution:



The vertex is at $(-2, -6)$.

By studying the previous example, it is possible to predict where the graphs will be. Each of the preceding graphs is congruent in shape to the graph of $y = x^2$. To obtain the previous graphs, you translate the graph of $y = x^2$ vertically and horizontally. The coordinates of the vertex can be identified by examining the equation of the function.



The graph of $y = (x - h)^2 + k$ can be sketched by translating the graph of $y = x^2$ k units vertically and h units horizontally. The coordinates of the vertex are at (h, k) .

When identifying the coordinates of the vertex, you must be careful to write down the correct values. Look at the following examples.

For $y = (x - 3)^2 + 2$, the vertex is at $(3, 2)$.

For $y = (x + 5)^2 + 4$, the vertex is at $(-5, 4)$.

For $y = (x + 4)^2 - 7$, the vertex is at $(-4, -7)$.

The graph of $y = (x - 5)^2 + 3$ is the graph of $y = x^2$ translated five units to the right and three units up. The coordinates of the vertex are $(5, 3)$. If you know the coordinates of the vertex, you should be able to describe how the graph of $y = x^2$ was translated.

Look at the following examples.

Example 9

For the quadratic function $y = (x + 1)^2 - 3$, do the following.

- Identify the coordinates of the vertex.

Solution:

The vertex is at $(-1, -3)$.

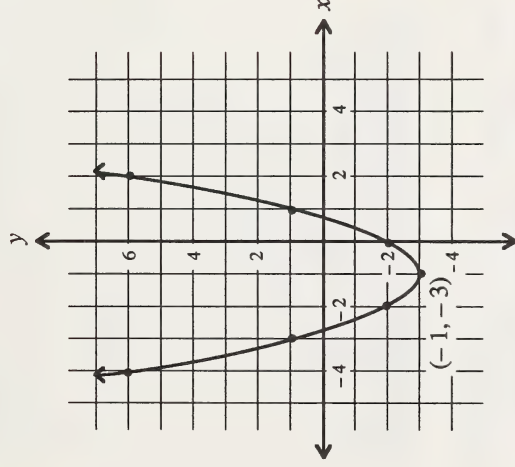
- Describe how the graph of $y = x^2$ was translated.

Solution:

The graph of $y = x^2$ was translated one unit to the left and three units downward.

- Sketch the graph.

Solution:



Example 10

For the quadratic function $y = (x - 2)^2 + 3$, do the following.

- Identify the coordinates of the vertex.

Solution:

The vertex is at $(2, 3)$.

$(-1, -3)$

A negative x -coordinate means the reference parabola was translated to the left.

A negative y -coordinate means the reference parabola was translated downward.

$(2, 3)$

A positive x -coordinate means the reference parabola was translated to the right.

A positive y -coordinate means the reference parabola was translated upward.

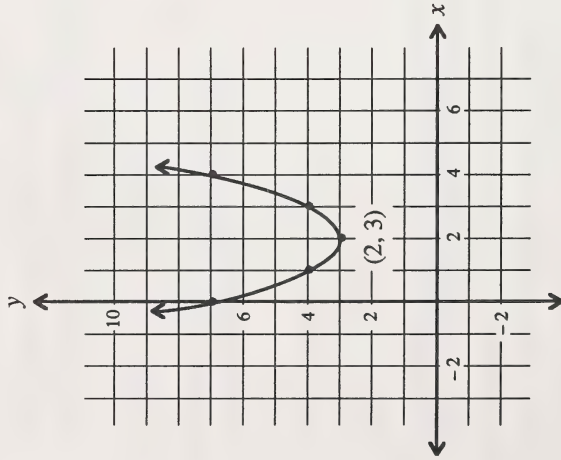
- Describe how the graph of $y = x^2$ was translated.

Solution:

The graph of $y = x^2$ was translated two units to the right and three units upward.

- Sketch the graph.

Solution:



The previous examples show that if you can identify the coordinates of the vertex of a parabola, you can sketch the parabola by translating the reference parabola.

You have been graphing parabolas that have the same shape as the reference parabola. However, some parabolas do not all have the same shape. Some parabolas are wider than others and some are narrower. You will now learn how to predict the shape of a parabola by just looking at the equation of the function. Follow the next example and see if you can make a prediction.

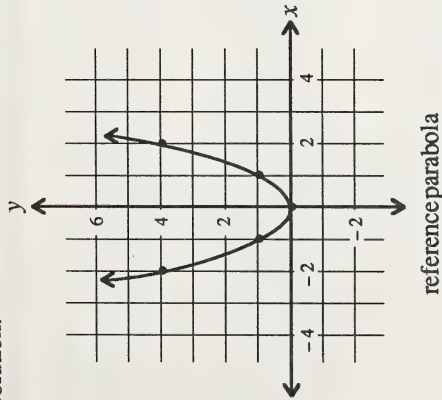
Example 11

Graph each quadratic function that follows and make a prediction about how the shape of the parabola and its equation are related.

- $y = x^2$ (reference function)

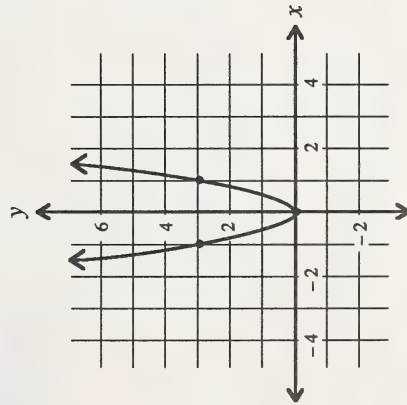
You may use a table of values, a computer, or a graphing calculator to sketch these graphs.

Solution:



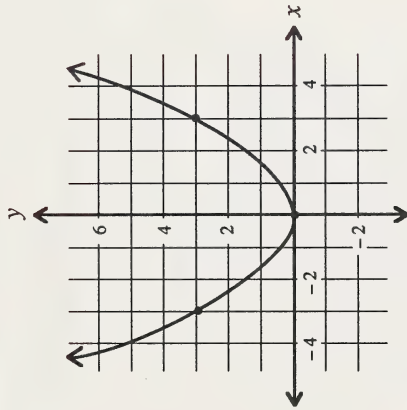
- $y = 3x^2$

Solution:



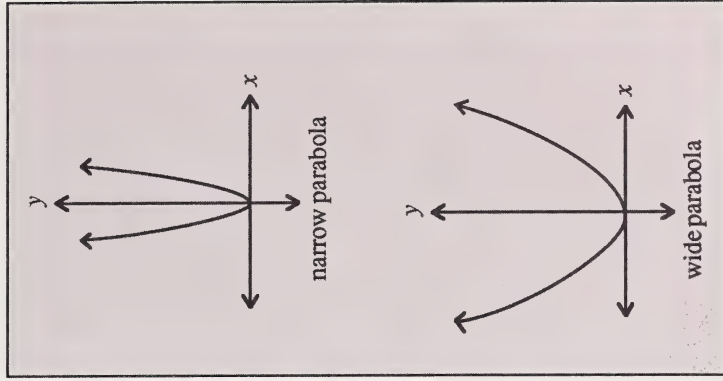
- $y = \frac{1}{3}x^2$

Solution:



The number in front of x^2 determines the shape of the parabola. The shape is determined as follows:

- If the number in front of x^2 is greater than one, the graph will be narrower than the graph of $y = x^2$.
- If the number in front of x^2 is less than one, the graph will be wider than the graph of $y = x^2$.



Note that the vertex for the three parabolas is the same, namely, $(0, 0)$. Numbers in front of the x^2 do not change the location of the vertex as horizontal and vertical translations do.

The previous information is summarized as follows.

For the quadratic function $y = ax^2$, the shape of the graph is determined by the value of a as indicated in the following:



- If $|a| = 1$, then the graph will be the reference parabola.
- If $|a| > 1$, then the graph will be narrower than the reference parabola.
- If $|a| < 1$, then the graph will be wider than the reference parabola.

Rather than using the terms wider or narrower to describe the shape of the parabola, you may use the terms **vertically stretched** to indicate narrower and **vertically compressed** to indicate wider.

The next example shows you how the width of a parabola is related to the number in front of the x^2 .

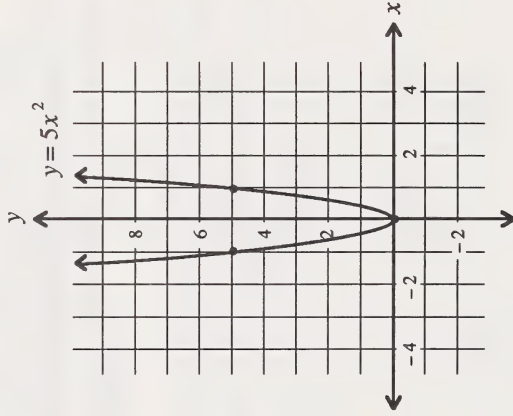
Example 12

Sketch the graphs of the following.

- $y = 5x^2$

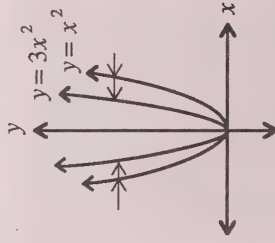
Solution:

Since $a = 5$, the graph of $y = 5x^2$ will be narrower than the graph of $y = x^2$.

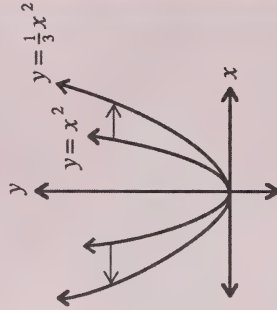


Note that a is any real number.

Note that $|a|$ means the absolute value of a .



A vertically stretched parabola becomes narrower.

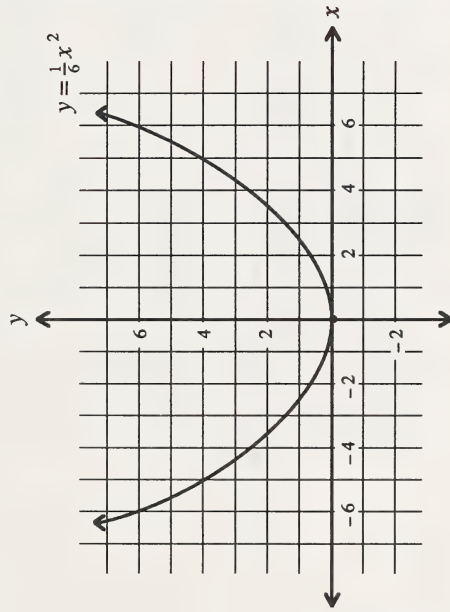


A vertically compressed parabola becomes wider.

- $y = \frac{1}{6}x^2$

Solution:

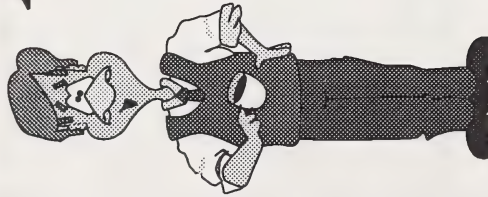
Since $a = \frac{1}{6}$, the graph of $y = \frac{1}{6}x^2$ will be wider than the graph of $y = x^2$.



Later in this unit you will be shown how to sketch a parabola approximately using the vertex and the value of a . At present you need to identify whether the graph will be wider or narrower than the graph of $y = x^2$.

At the beginning of this topic you were told that you would be studying parabolas that open upward or downward. You have studied parabolas that open upward. What causes a parabola to open downward? Can you predict whether a parabola will open downward? Can you predict whether a parabola will open downward or upward by just looking at the equation of the function?

To answer these questions, look at the next example.

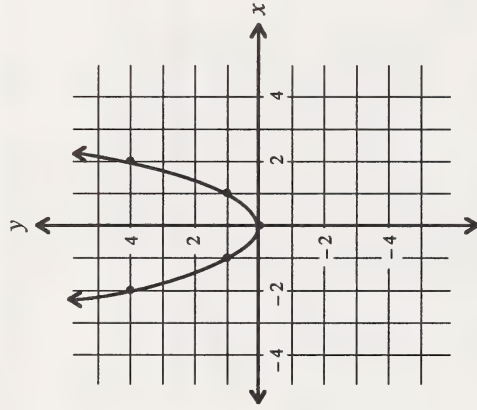


Example 13

Graph the following functions and make a statement about how you may predict whether a parabola will open upward or downward.

- $y = x^2$

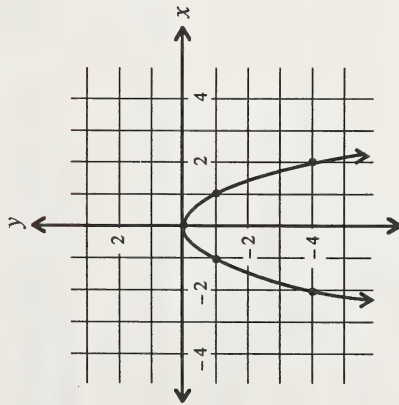
Solution:



Parabolas may also open sideways, but these parabolas are not functions. Since the focus of study in this unit is on quadratic functions, parabolas that open sideways will not be discussed.

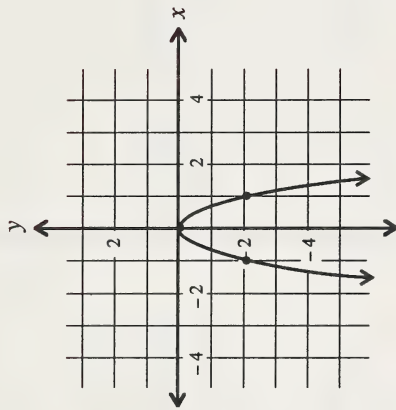
- $y = -x^2$

Solution:



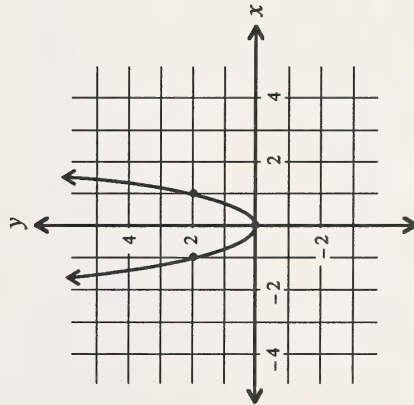
- $y = -2x^2$

Solution:



- $y = 2x^2$

Solution:



It appears that the sign of the number in front of the x^2 determines whether the parabola opens up or down. If the number in front of the x^2 is positive, the parabola opens upward. If the number in front of the x^2 is negative, the parabola opens downward.

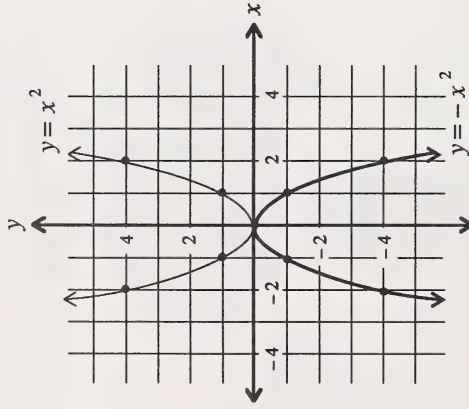


For the quadratic function $y = ax^2$, the parabola will do the following:

- open upward if $a > 0$ (a is positive)
- open downward if $a < 0$ (a is negative)

The graph of $y = -x^2$ is a reflection in the x -axis of the graph of $y = x^2$.

If you fold the graph paper along the x -axis, the graph of $y = x^2$ will coincide exactly with the graph of $y = -x^2$. The graph of $y = -x^2$ is a mirror reflection in the x -axis of the graph of $y = x^2$.



The graph of $y = -2x^2$ is a reflection in the x -axis of the graph of $y = 2x^2$.

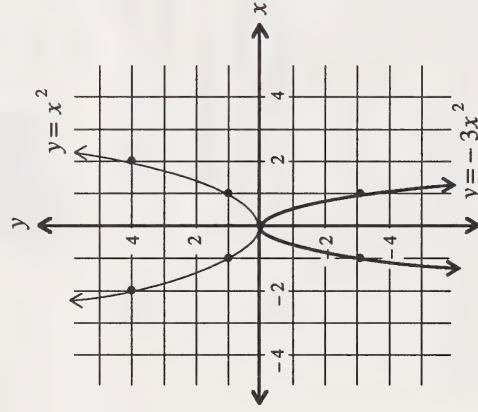
For the quadratic function $y = ax^2$, you can see that

- a determines the width of the parabola
- a determines the direction of opening

Example 14

Sketch the graphs of $y = x^2$ and $y = -3x^2$ and describe how the graph of $y = -3x^2$ can be obtained from the graph of $y = x^2$.

Solution:



The negative sign indicates that the graph opens downward. The 3 indicates that the graph is narrower than the graph of $y = x^2$.

The graph of $y = x^2$ opens upward. The reflection in the x -axis of $y = x^2$ is $y = -x^2$ which opens downward.

You know that the graph of the reference function $y = x^2$ is the reference parabola. Using this idea enables you to graph any quadratic function by inspecting the equation of the function. Every number in the equation tells you something about the location or the shape of the graph.

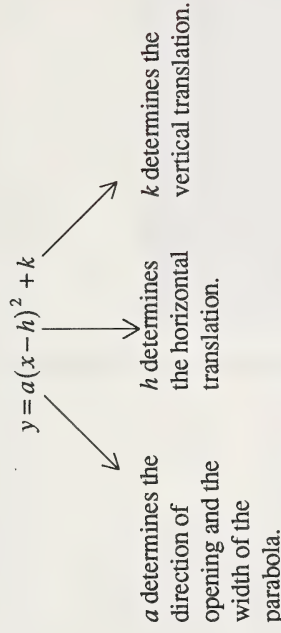
Before you start sketching quadratic functions by inspection, a brief summary of the ideas presented so far is given as follows.

For the reference function $y = x^2$, the vertex is at $(0, 0)$.

- The graph of $y = x^2 + k$ is a vertical translation of the graph of $y = x^2$ by k units. The vertex is at $(0, k)$.
- The graph of $y = (x - h)^2$ is a horizontal translation of the graph of $y = x^2$ by h units. The vertex is at $(h, 0)$.

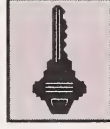
- The graph of $y = ax^2$ can be obtained from the graph of $y = x^2$ by reflecting in the x -axis (if a is negative) and changing the width of the parabola according to the value of $|a|$. The vertex is at $(0, 0)$.

The previous three forms of quadratic functions can be combined into the one general form shown.



Since the graph of $y = x^2$ was translated h units horizontally and k units vertically, the vertex must be at (h, k) .

For the quadratic function $y = a(x - h)^2 + k$, the following is true:



- The vertex of the parabola is at (h, k) .
- The direction of opening and the width of the parabola are determined by the value of a .

It is important that you are able to identify the coordinates of the vertex. Once the coordinates of the vertex are known, it is relatively simple to graph the function. The following example shows you how to identify the coordinates of the vertex.

Example 15

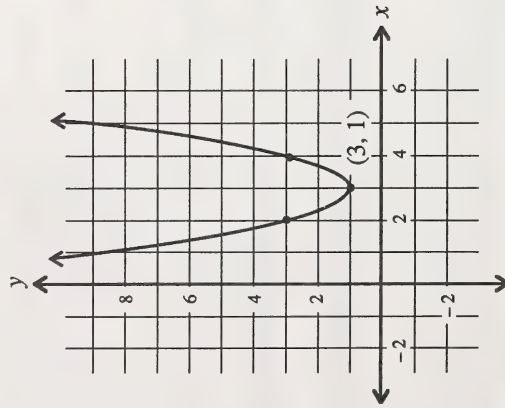
For each quadratic function, identify the coordinates of the vertex and sketch the graph.

- $y = 2(x - 3)^2 + 1$

Solution:

The vertex is at $(3, 1)$.

Since $a = 2$, the graph will open upward and it will be narrower than the graph of $y = x^2$.

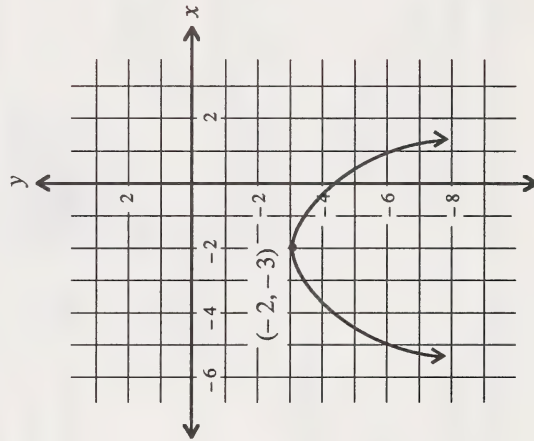


- $y = -\frac{1}{3}(x + 2)^2 - 3$

Solution:

The vertex is at $(-2, -3)$.

Since $a = -\frac{1}{3}$, the graph will open downward and it will be wider than the graph of $y = x^2$.



The sign inside the parentheses is important in determining the vertex.

For $y = 2(x - 3)^2 + 1$, the vertex is at $(3, 1)$ since this matches the general form of $y = a(x - h)^2 + k$. In this case, $h = 3$ and $k = 1$.

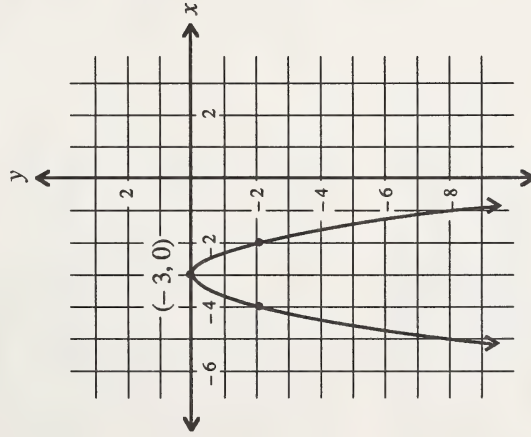
For $y = -\frac{1}{3}(x + 2)^2 - 3$, the vertex is at $(-2, -3)$ since you must rewrite the function as $y = -\frac{1}{3}[x - (-2)]^2 - 3$ to match the general form. In this case, $h = -2$ and $k = -3$.

- $y = -2(x + 3)^2$

Solution:

The vertex is at $(-3, 0)$.

Since $a = -2$, the graph will open downward and it will be narrower than the graph of $y = x^2$.



The shape of the parabola can be sketched by considering the value of a . Locate the vertex and then use the value of a to determine how wide or narrow the parabola is. The next example shows you how this is done.

Example 16

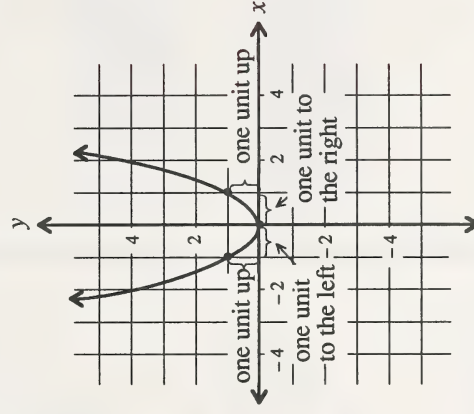
For each quadratic function, identify the vertex and then use the value of a to approximate the shape of the parabola.

- $y = x^2$

Solution:

The vertex is at $(0, 0)$.

Note that a has a value of one ($y = 1x^2$). The shape of the parabola can be approximated by moving one unit to the right and one unit to the left of the vertex. Then move up one unit on the right and the left.



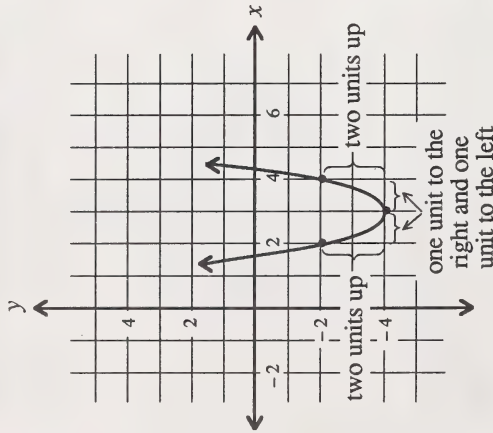
Since $a = 1$, move up one unit.

- $y = 2(x - 3)^2 - 4$

Solution:

The vertex is at $(3, -4)$.

Note that a has a value of two. To approximate the shape of the parabola, start at the vertex and move one unit to the right and one unit to the left. Now move up two units on the left and the right.



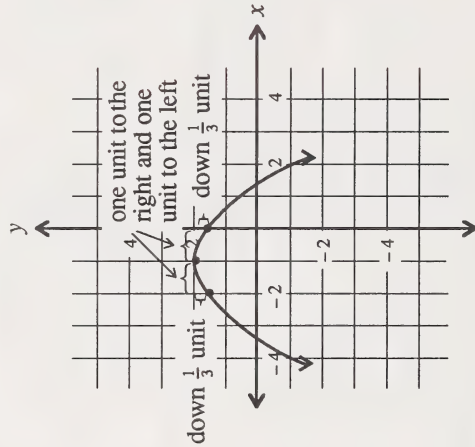
Since $a = 2$, move up two units.

- $y = -\frac{1}{3}(x + 1)^2 + 2$

Solution:

The vertex is at $(-1, 2)$.

Note that a has a value of $-\frac{1}{3}$. To approximate the shape of the parabola, move one unit to the right and one unit to the left. Now move down one-third unit on the left and the right.



Since $a = -\frac{1}{3}$, move down $\frac{1}{3}$ unit.

The previous method gives you three points through which you may sketch the parabola.

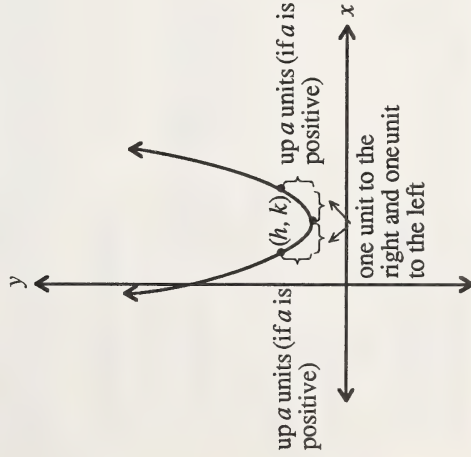
The negative sign means that the graph will move and open downward.



To approximate the shape of the parabola defined by

$y = a(x - h)^2 + k$, start at the vertex (h, k) and move one unit to the right and one unit to the left. Now move a units up (if a is positive) or a units down (if a is negative). Sketch the parabola through the three points.

The graph of $y = a(x - h)^2 + k$ is shown.



Every quadratic function can be written in the form $y = a(x - h)^2 + k$. From this form you can identify the coordinates of the vertex, the direction of opening, and the width of the parabola. Frequently, quadratic functions are not given in this form and before you can identify the coordinates of the vertex, you should write them in this form. The next example shows how this is done.

Example 17

Write each quadratic function in the form

$y = a(x - h)^2 + k$ and name the coordinates of the vertex.

- $y = 2x^2$

Solution:

The function $y = 2x^2$ can be written as

$$y = 2(x - 0)^2 + 0. \text{ The vertex is at } (0, 0).$$

- $y = -3x^2 + 2$

Solution:

The function $y = -3x^2 + 2$ can be written as $y = -3(x - 0)^2 + 2$. The vertex is at $(0, 2)$.

The results obtain here are consistent with the results obtained earlier in this topic.

- $y = \frac{1}{2}(x-4)^2$

Solution:

The function $y = \frac{1}{2}(x-4)^2$ can be written as $y = \frac{1}{2}(x-4)^2 + 0$.

The vertex is at $(4, 0)$.

- $y = 2(x+1)^2$

Solution:

The function $y = 2(x+1)^2$ can be written as

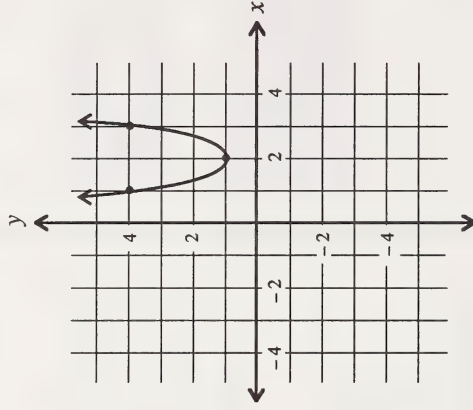
$$y = 2(x - (-1))^2 + 0.$$

The vertex is at $(-1, 0)$.

If you can identify the coordinates of the vertex from the equation of a quadratic function, you should be able to sketch the graph of this function. Conversely, if you can identify the coordinates of the vertex from the graph, you should be able to write the function that describes the graph. The next example shows you how to do this.

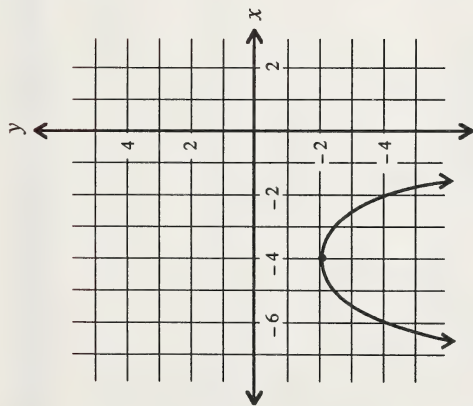
Example 18

For each parabola, identify the coordinates of the vertex, approximate the value of a , and write the equation of the function.



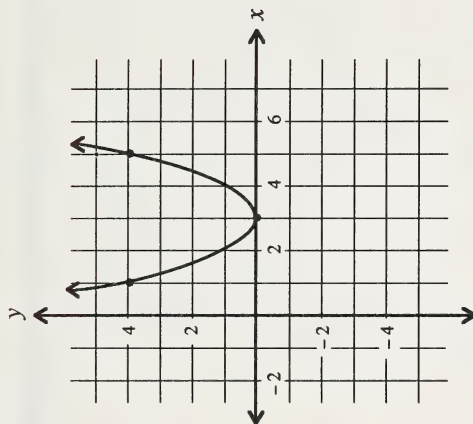
Solution:

Since the vertex is at $(2, 1)$ and $a = 3$, the equation of the function is $y = 3(x-2)^2 + 1$.



Solution:

Since the vertex is at $(-4, -2)$ and $a = -\frac{1}{2}$,
the equation of the function is
 $y = -\frac{1}{2}(x+4)^2 - 2$.



Solution:

Since the vertex is at $(3, 0)$ and $a = 1$, the
equation of the function is
 $y = 1(x-3)^2 + 0$ or $y = (x-3)^2$.

Notice how the coordinates of the vertex and
the equation of the function are related.

Complete the following questions involving transformations. You may use a computer program or a graphing calculator to help you answer these questions. Graph paper is provided in **Appendix B**.

20. Describe how you may obtain the graph of the following.

a. $y = (x-3)^2 + 2$ from the graph of $y = x^2$

b. $y = (x+8)^2 - 5$ from the graph of $y = x^2$

21. Write the coordinates of the vertex for each function that follows.

a. $y = (x-8)^2 - 3$

b. $y = (x+7)^2 + 14$

c. $y = (x-h)^2 + k$

22. By translating the graph of $y = x^2$, sketch the graph of any two functions that follow. Identify the coordinates of the vertex.

a. $y = (x+1)^2 - 1$

b. $y = (x-2)^2 + 3$

c. $y = (x+3)^2 + 2$

d. $y = (x-1)^2 - 2$

23. Explain the difference between the graphs of $y = 3x^2$ and $y = -3x^2$.

24. In what two ways does the value of a affect the graph of $y = ax^2$?

25. Describe the difference between the graphs of the following.

a. $y = 8x^2$ and $y = x^2$

b. $y = \frac{1}{5}x^2$ and $y = x^2$

c. $y = -3x^2$ and $y = x^2$

d. $y = -\frac{1}{4}x^2$ and $y = x^2$

26. Sketch any two of the quadratic functions that follow.

a. $y = 2(x-3)^2 + 4$

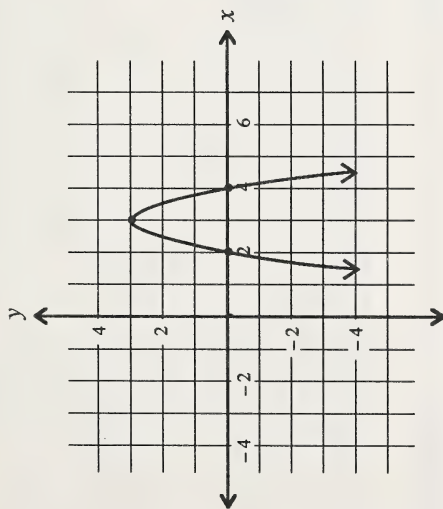
b. $y = -3(x+2)^2 - 1$

c. $y = \frac{1}{2}(x-1)^2 - 3$

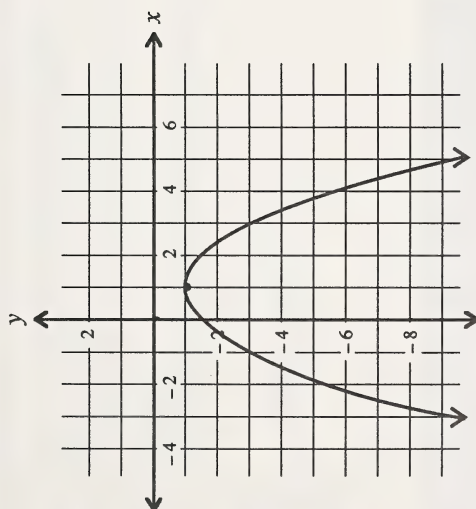
d. $y = -\frac{1}{4}(x+1)^2 + 2$

27. Write the equation that describes each of the following parabolas.

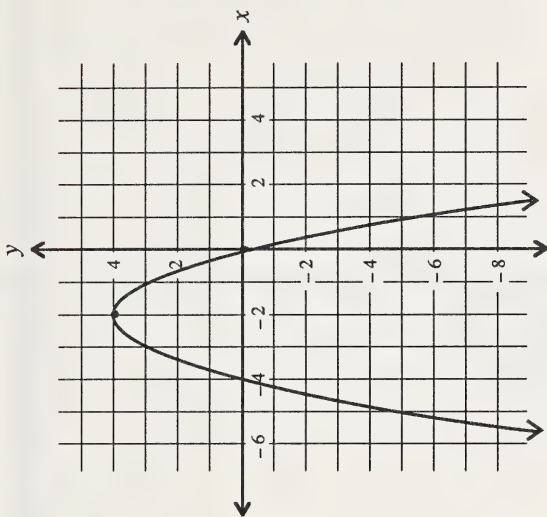
a.



b.



c.



For solutions to Activity 1, turn to Appendix A,
Topic 1.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.



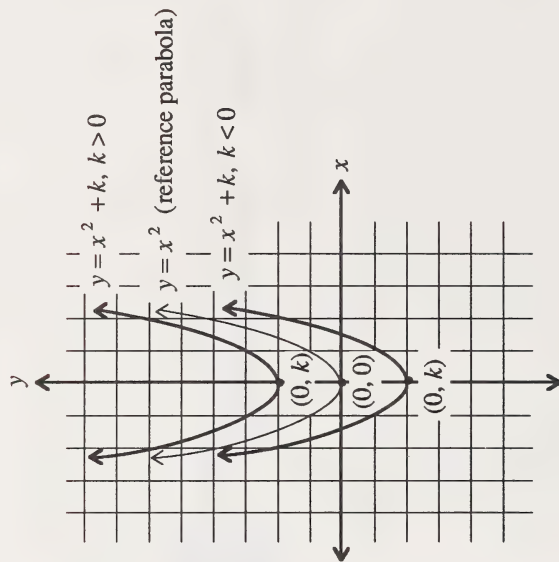
Extra Help

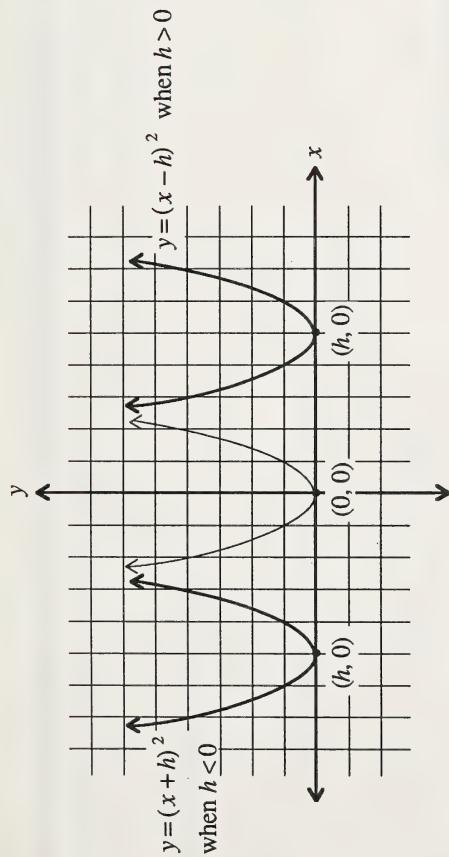
The two most important things to determine when graphing quadratic functions are as follows:

- the coordinates of the vertex
- the shape of the parabola and the direction of opening

If you have trouble with the previous two concepts, then you should go over this section.

The graph of $y = x^2 + k$ is the same as the graph of $y = x^2$ translated k units vertically upward if k is greater than zero or translated k units vertically downward if k is less than zero. The vertex is at $(0, k)$. It is above the x -axis if k is greater than zero and below the x -axis if k is less than zero.





Examine the relationship between the sign of h and the translation that takes place.

For $y = (x - 5)^2$, $h = 5$. Since h is positive, the translation is to the right.

For $y = (x + 5)^2$, rewrite the function as $y = [x - (-5)]^2$; therefore, $h = -5$. Since h is negative, the translation is to the left.

The emphasis in this section will be on identifying the correct horizontal translation.



In general for $y = (x - h)^2 + k$, if you replace h and k with real numbers, the coordinates of the vertex can be identified.

- $y = (x - 5)^2 + 3$ The vertex is at $(5, 3)$.
- $y = (x + 5)^2 + 3$ The vertex is at $(-5, 3)$.
- $y = (x - 5)^2 - 3$ The vertex is at $(5, -3)$.
- $y = (x + 5)^2 - 3$ The vertex is at $(-5, -3)$.

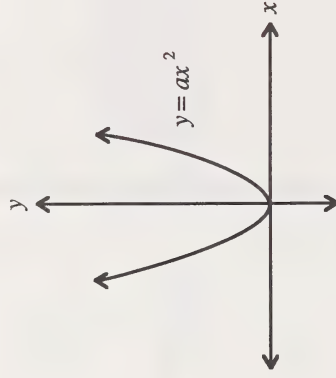
To see how you may identify the coordinates of the vertex, follow these examples.

- $y = x^2$ The vertex is at $(0, 0)$ since the relation is $y = (x - 0)^2 + 0$.
- $y = x^2 - 5$ The vertex is at $(0, -5)$ since the relation is $y = (x - 0)^2 - 5$.
- $y = (x - 3)^2$ The vertex is at $(3, 0)$ since the relation is $y = (x - 3)^2 + 0$.
- $y = (x + 7)^2$ The vertex is at $(-7, 0)$ since the relation is $y = (x + 7)^2 + 0$.
- $y = (x + 2)^2 - 3$ The vertex is at $(-2, -3)$.

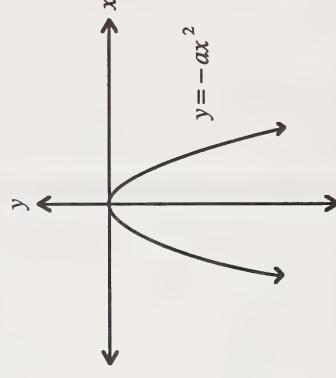
Now that you can identify the coordinates of the vertex, you should see how the value of a determines the shape and the location of the parabola.

For the parabola defined by $y = ax^2$, the following is true.

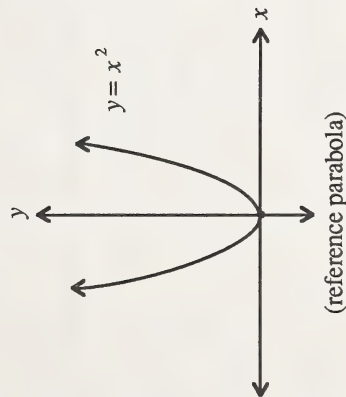
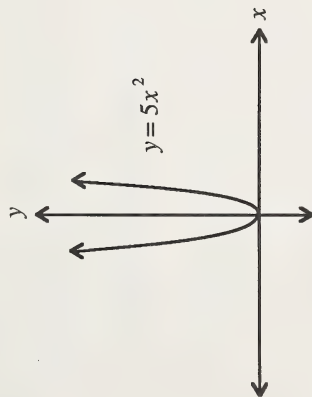
- a is positive when the parabola opens upward.



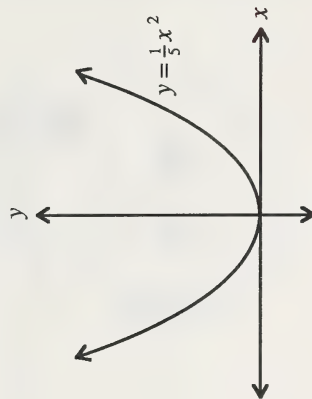
- a is negative when the parabola opens downward.



- If the $|a|$ is greater than 1 (for example, $|2|$, $|-2|$, $|3|$, $|-3|$, $|4|$, $|-4|$), the parabola will be narrower than the reference parabola.



- If the $|a|$ is smaller than 1 (for example, $|\frac{1}{2}|$, $|\frac{1}{3}|$, $|\frac{2}{3}|$, $|\frac{2}{5}|$, $|\frac{1}{4}|$, $|\frac{1}{4}|$), the parabola will be wider than the reference parabola.



Look at the following examples to see how the value of a influences the width and the direction of the opening.

- If $y = 2x^2$, then $a = 2$. Therefore, the parabola opens upward and is narrower than $y = x^2$.
- If $y = -\frac{1}{2}x^2$, then $a = -\frac{1}{2}$. Therefore, the parabola opens downward and is wider than $y = x^2$.
- If $y = -5(x-4)^2 + 6$, then $a = -5$. Therefore, the parabola opens downward and is narrower than $y = x^2$.
- If $y = \frac{1}{4}(x+2)^2 - 3$, then $a = \frac{1}{4}$. Therefore, the parabola opens upward and is wider than $y = x^2$.

Do the following exercises to see how well you understand the previous material.

1. Identify the vertex for each function.

a. $y = x^2$

b. $y = -3x^2$

c. $y = x^2 + 2$

d. $y = 3x^2 - 2$

e. $y = (x-5)^2$

f. $y = -4(x+2)^2$

g. $y = \frac{2}{3}(x-1)^2$

h. $y = (x+2)^2 - 3$

i. $y = 4(x-7)^2 + 19$

j. $y = 14(x+19)^2 - 72$

k. $y = 172(x-475)^2 - 189$

l. $y = -\frac{1}{7}(x+59)^2$

2. State whether the graph of each function opens upward or downward and state whether the parabola will be wider or narrower than the reference parabola.

a. $y = -2x^2$

b. $y = \frac{1}{3}x^2 - 7$

c. $y = 5(x-3)^2 + 7$

d. $y = -\frac{3}{7}(x+2)^2 - 4$

e. $y = -(x+3)^2$

f. $y = -5x^2 - 7$

g. $y = \frac{3}{2}(x-2)^2$



For solutions to Extra Help, turn to Appendix A, Topic 1.





Extensions

If you found sketching quadratic functions interesting and would like to improve your skills, here are more difficult questions you may try.

1. Where would the parabola of $y = x^2 + 565$ be located?
2. Where would the parabola of $y = (x - 10^4)^2$ be located?
3. Where would the parabola of $y = (x + 4^3)^2 - 5^5$ be located?
4. Write the coordinates of the vertex for
 - a. the parabola in question 1
 - b. the parabola in question 2
 - c. the parabola in question 3
5. What would the parabola defined by $y = 1000(x - 2)^2 + 6$ look like?
6. What would the parabola defined by $y = -5^{-4}(x + 2)^2 - 3$ look like?



For solutions to **Extensions**, turn to **Appendix A**,
Topic 1.

Topic 2 Completing the Square



Introduction

The quadratic function $y = a(x - h)^2 + k$ can easily be graphed by identifying the coordinates of the vertex and using the value of a to approximate the shape of the parabola. Another form for a quadratic function is $y = ax^2 + bx + c$. This form is not as useful because the coordinates of the vertex cannot readily be identified. In this topic you will be changing the form $y = ax^2 + bx + c$ into the form $y = a(x - h)^2 + k$ and identifying the coordinates of the vertex.



What Lies Ahead

Throughout the topic you will learn to

1. change quadratic functions in the form $y = ax^2 + bx + c$ to the form $y = a(x - h)^2 + k$ by completing the square

Now that you know what to expect, turn the page to begin your study of completing the square.



Exploring Topic 2

Activity 1



Change quadratic functions
in the form $y = ax^2 + bx + c$
to the form
 $y = a(x-h)^2 + k$ by
completing the square.

Which method for graphing parabolas do you find easier?

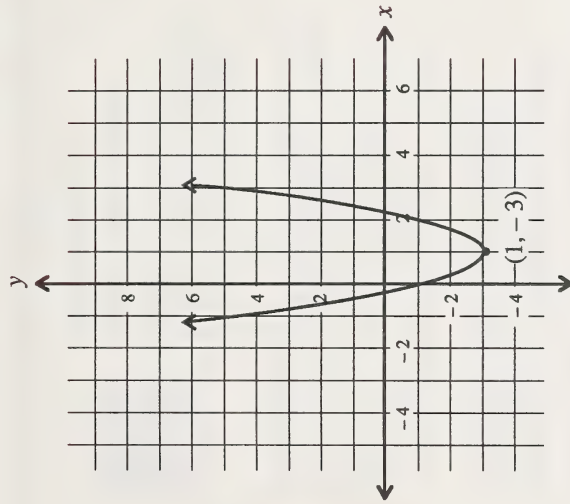
Look at the following example.

Example 1

- Graph $y = 2(x-1)^2 - 3$.

Solution:

The vertex is at $(1, -3)$. Since $a = 2$, the graph opens upward and is narrower than the reference parabola.

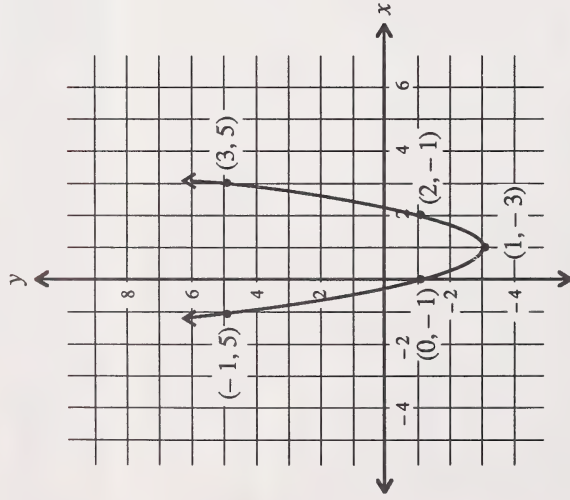


- Graph $y = 2x^2 - 4x - 1$.

Solution:

x	-2	-1	0	1	2	3	4
y	15	5	-1	-3	-1	5	15

You cannot identify the vertex from $y = 2x^2 - 4x - 1$; therefore, to graph the function, you need to complete a table of values and plot the ordered pairs.



You should draw two conclusions from the previous example.

- It is easier to graph a parabola if the coordinates of the vertex are known.
- The parabolas are congruent no matter which method is used.

Since the two parabolas are congruent, the two functions are equivalent.

The function $y = 2(x-1)^2 - 3$ must be equivalent to the function $y = 2x^2 - 4x - 1$.

$$y = 2(x-1)^2 - 3$$

$$y = 2(x^2 - 2x + 1) - 3$$

$$y = 2x^2 - 4x + 2 - 3$$

$$y = 2x^2 - 4x - 1$$

It was relatively simple to change

$y = 2(x-1)^2 - 3$ into the form

$y = 2x^2 - 4x - 1$. What about the reverse

process? Can $y = 2x^2 - 4x - 1$ be changed to the form $y = 2(x-1)^2 - 3$? If it can be changed to this form, then graphing would be much simpler. You will be shown how to perform this reverse process in this topic.

At the beginning of this unit, it was stated that functions like $y = x^2$, $y = x^2 + 2$, and $y = 2x^2$ belong to a special group of functions called **quadratic functions**. It is time to give a more formal definition of what a quadratic function is.

A **quadratic function** is any function that can be written in the form $y = ax^2 + bx + c$, where a , b , and c are real numbers and $a \neq 0$.



$$\begin{aligned}(x-1)^2 &= (x-1)(x-1) \\ &= x^2 - 2x + 1\end{aligned}$$

If $a = 0$, then $y = ax^2 + bx + c$ would become $y = bx + c$, which is not quadratic.

The form $y = ax^2 + bx + c$ is said to be the standard form for the equation of a quadratic function.

- $y = 3x - 5x^2 + 7$

Solution:

The letters a , b , and c represent the following numerical values:

- $a = -5x^2 + 3x + 7$
- $a = -5, b = 3, c = 7$
- $7 - 2x + 5x^2 = y$

Example 2

Write each function in standard form and identify the values of a , b , and c .

- $y - 3 = 2x^2 + 5x$

Solution:

$$y = 2x^2 + 5x + 3$$
$$a = 2, b = 5, c = 3$$

- $y - 2x^2 = 0$

Solution:

$$y = 2x^2$$
$$a = 2, b = 0, c = 0$$

Solution:

$$y = 2x^2 + 5$$
$$a = 2, b = 0, c = 5$$

The word **quadratic** implies a degree of two. All quadratic functions are of second degree.

In the last topic you were shown how to graph quadratic functions in the form $y = a(x-h)^2 + k$. This form is called the **completed square form** for the equation of a quadratic function. Recall that a determines the direction of the opening and the width of the parabola, and (h, k) are the coordinates of the vertex.



The equation for a quadratic function has two forms. These forms are as follows:

- $y = ax^2 + bx + c$ (standard form)
- $y = a(x-h)^2 + k$ (completed square form)

For graphing purposes, the completed square form is more useful since it allows you to identify the coordinates of the vertex of the parabola.

Before you are shown how to change the standard form of a quadratic function into its equivalent completed square form, you should review some factoring skills.

Recall that some trinomials are factorable into a binomial that is squared. For example, $x^2 + 6x + 9$ is factorable into $(x+3)^2$. Trinomials like this are called perfect square trinomials.

Can you tell by just looking at a trinomial whether it is a perfect square trinomial?

Following are some perfect square trinomials. Examine the number in front of the x and the constant term. Is there a relationship between the two? If you know one of these numbers, can you tell what the other has to be?

$$\begin{array}{c} x^2 + 6x + 9 \\ \uparrow \qquad \qquad \uparrow \\ \left(\frac{1}{2} \times 6\right)^2 \\ = (3)^2 \\ = 9 \end{array}$$

$$\begin{array}{c} x^2 + 10x + 25 \\ \uparrow \qquad \qquad \uparrow \\ \left(\frac{1}{2} \times 10\right)^2 \\ = (5)^2 \\ = 25 \end{array}$$

$$\begin{array}{c} x^2 - 8x + 16 \\ \uparrow \qquad \qquad \uparrow \\ \left(\frac{1}{2} \times -8\right)^2 \\ = (-4)^2 \\ = 16 \end{array}$$

All three perfect square trinomials exhibit the same pattern, namely, that the square of $\frac{1}{2}$ times the coefficient of x will give the constant term. This pattern is true for all perfect square trinomials of the form $x^2 + bx + c$.



If $x^2 + bx + c$ is a perfect square trinomial then $\left(\frac{1}{2} \times b\right)^2 = c$.

- $x^2 + \frac{3}{5}x + c$

Solution:

$$\left(\frac{1}{2} \times \frac{3}{5}\right)^2 = c$$

$$\left(\frac{3}{10}\right)^2 = c$$

$$\frac{9}{100} = c$$

If you know the value of b , you can tell what c must be in order for the trinomial to be a perfect square trinomial. Follow the next example which shows you how to do this calculation.

Example 3

The first two terms of a trinomial are given. Determine what the constant term must be in order for the trinomial to be a perfect square trinomial.

- $x^2 + 4x + c$

Solution:

$$\left(\frac{1}{2} \times 4\right)^2 = c$$

$$(2)^2 = c$$

$$4 = c$$

- $x^2 - 12x + c$

Solution:

$$\left(\frac{1}{2} \times -12\right)^2 = c$$

$$(-6)^2 = c$$

$$36 = c$$

Now that you can create your own perfect square trinomials, you should learn how to factor them by inspection. Look at the next example which shows you how to factor perfect square trinomials.

Example 4

Factor the following perfect square trinomials.

- $x^2 + 6x + 9$

Solution:

$$x^2 + 6x + 9$$

Diagram illustrating the factoring process for $x^2 + 6x + 9$:

- x^2 and 9 are identified as perfect squares: $\sqrt{x^2} = x$ and $\sqrt{9} = 3$.
- The middle term $6x$ is identified as $2 \times x \times 3$.
- The trinomial is factored as $(x + 3)^2$.

- $x^2 - 8x + 16$

Solution:

$$\begin{array}{c}
 x^2 - 8x + 16 \\
 \swarrow \quad \searrow \quad \downarrow \\
 \sqrt{x^2} = x \quad \sqrt{16} = 4 \\
 \quad \quad \quad \downarrow \quad \downarrow \\
 \quad \quad \quad (x-4)^2
 \end{array}$$

- $x^2 + \frac{3}{5}x + \frac{9}{100}$

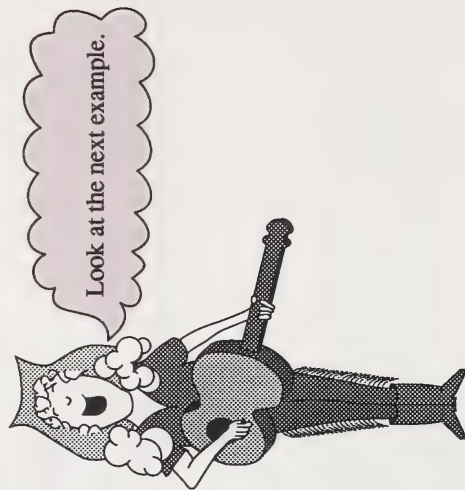
Solution:

$$\begin{array}{c}
 x^2 + \frac{3}{5}x + \frac{9}{100} \\
 \swarrow \quad \searrow \quad \downarrow \\
 \sqrt{x^2} = x \quad \sqrt{\frac{9}{100}} = \frac{3}{10} \\
 \quad \quad \quad \downarrow \quad \downarrow \\
 \quad \quad \quad \left(x + \frac{3}{10}\right)^2
 \end{array}$$

Note how the sign inside the binomial is determined.

$$\begin{array}{c}
 x^2 - 12x + 36 \\
 \downarrow \\
 (x-6)^2
 \end{array}
 \qquad
 \begin{array}{c}
 x^2 + 14x + 49 \\
 \downarrow \\
 (x+7)^2
 \end{array}$$

You should now have the skills required to convert the standard form of a quadratic function to the completed square form. This process is called **completing the square**. The emphasis of this topic is on this process and it is important that you understand every step.



Example 5

Change $y = x^2 + 6x$ into completed square form and identify the vertex.

Solution:

$$y = x^2 + 6x$$

Determine what the last term must be in order to have a perfect square trinomial.

$$\left(\frac{1}{2} \times 6\right)^2 = (3)^2 \\ = 9$$

The last term must be 9.

Add 9 to complete the square and subtract 9 to keep the function in balance.

$$y = x^2 + 6x + 9 - 9$$

The first three terms form a perfect square trinomial; therefore, place parentheses around them.

$$y = (x^2 + 6x + 9) - 9$$

Factor the perfect square trinomial.

$$y = (x + 3)^2 - 9$$

The completed square form is

$y = (x + 3)^2 - 9$ and the coordinates of the vertex are at $(-3, -9)$.

Example 6

Change $y = x^2 + 10x + 28$ into completed square form and identify the coordinates of the vertex.

Solution:

Place parentheses around the first two terms and leave a space.

$$y = (x^2 + 10x) + 28$$

Determine what the last term in the parentheses must be in order to have a perfect square trinomial.

$$\left(\frac{1}{2} \times 10\right)^2 = (5)^2 \\ = 25$$

The last term must be 25.

The two terms inside the brackets must be the x^2 term and the x term.

Add 25 inside the parentheses and subtract 25 outside the parentheses. This keeps the function in balance.

$$y = (x^2 + 10x + 25) + 28 - 25$$

Factor the perfect square trinomial inside the parentheses.

$$y = (x + 5)^2 + 3$$

The completed square form is

$$y = (x + 5)^2 + 3. \text{ The vertex is at } (-5, 3).$$

The next example involves fractions. Follow it closely.

Example 7

Find the coordinates of the vertex for

$$y = x^2 - 5x + 2. \text{ (Put into completed square form.)}$$

Solution:

$$y = (x^2 - 5x) + 2$$

Complete the square inside the parentheses.

$$\begin{aligned} \left(\frac{1}{2} \times -5\right)^2 &= \left(\frac{-5}{2}\right)^2 \\ &= \frac{25}{4} \end{aligned}$$

Add $\frac{25}{4}$ inside the parentheses and subtract $\frac{25}{4}$ outside the parentheses.

$$y = \left(x^2 - 5x + \frac{25}{4}\right) + 2 - \frac{25}{4}$$

Factor.

$$\begin{aligned} y &= \left(x - \frac{5}{2}\right)^2 + \frac{8}{4} - \frac{25}{4} \\ y &= \left(x - \frac{5}{2}\right)^2 - \frac{17}{4} \end{aligned}$$

The vertex is at $\left(\frac{5}{2}, -\frac{17}{4}\right)$.

In the previous three examples, the coefficient of x^2 was one. How do you complete the square if the coefficient is a number other than one?

Look at the next example.

Example 8

Find the completed square form and the vertex for $y = 3x^2 - 12x + 17$.

Solution:

Group the first two terms.

$$y = (3x^2 - 12x) + 17$$

Factor the coefficient of x^2 from the first two terms.

$$y = 3(x^2 - 4x) + 17$$

Complete the square inside the parentheses.

$$\begin{aligned} \left(\frac{1}{2} \times -4\right)^2 &= (-2)^2 \\ &= 4 \end{aligned}$$

Add 4 inside the parentheses. To keep the function in balance, you must subtract $3 \times 4 = 12$ outside the parentheses.

$$y = 3(x^2 - 4x + 4) + 17 - 12$$

$3 \times 4 = 12$

Factor.

$$y = 3(x - 2)^2 + 5$$

The vertex is at $(2, 5)$.

The next example involves fractions.

Example 9

Find the completed square form and the vertex for $y = 2x^2 + 5x - 1$.

Solution:

Place parentheses around the two terms containing x^2 and x .

$$y = (2x^2 + 5x) - 1$$

Factor 2 from the first two terms.

$$y = 2\left(x^2 + \frac{5x}{2}\right) - 1$$

Complete the square inside the parentheses.

$$\begin{aligned} \left(\frac{1}{2} \times \frac{5}{2}\right)^2 &= \left(\frac{5}{4}\right)^2 \\ &= \frac{25}{16} \end{aligned}$$

In order to complete the square, the coefficient of x^2 must be one. This is why you factor out a 3 from $3x^2 - 12x$. Three factored from $3x^2$ gives x^2 .

To factor 2 from $2x^2 + 5x$, you can divide $2x^2 + 5x$ by 2. For example, $\frac{2x^2}{2} + \frac{5x}{2} = x^2 + \frac{5}{2}x$. Do not forget to put 2 in front of the parentheses.

In Example 8, adding 4 inside the parentheses results in adding a total of 12; therefore, 12 has to be subtracted outside the parentheses.

Adding $\frac{25}{16}$ inside the parentheses results in a total of $(2 \times \frac{25}{16})$ being added. Therefore, $(2 \times \frac{25}{16})$ has to be subtracted outside the parentheses in order for the function to remain in balance.

$$\begin{aligned}y &= 2\left(x^2 + \frac{5}{2}x + \frac{25}{16}\right) - 1 - 2\left(\frac{25}{16}\right) \\y &= 2\left(x^2 + \frac{5}{2}x + \frac{25}{16}\right) - \frac{8}{8} - \frac{25}{8} \\y &= 2\left(x^2 + \frac{5}{2}x + \frac{25}{16}\right) - \frac{33}{8}\end{aligned}$$

Factor and simplify.

$$y = 2\left(x + \frac{5}{4}\right)^2 - \frac{33}{8}$$

The completed square form is

$$y = 2\left(x + \frac{5}{4}\right)^2 - \frac{33}{8} \text{ and the vertex is at } \left(-\frac{5}{4}, -\frac{33}{8}\right).$$

The next example involves a negative number in front of the x^2 .

Example 10

Find the vertex for $y = -3x^2 + 42x - 135$.

Solution:

Change to the completed square form.

Place parentheses around the first two terms.

$$y = (-3x^2 + 42x) - 135$$

Factor -3 from the first two terms.

$$y = -3(x^2 - 14x) - 135$$

Complete the square inside the parentheses.

$$\begin{aligned}\left(\frac{1}{2} \times -14\right)^2 &= (-7)^2 \\ &= 49\end{aligned}$$

Recall: If the value in front of the x^2 is negative, the parabola opens down.

To factor -3 from $-3x^2 + 42x$, you can divide $-3x^2 + 42x$ by -3 .

For example,

$$\frac{-3x^2 + 42x}{-3} = x^2 - 14x$$

Do not forget to put -3 in front of the parentheses.

Adding 49 inside the parentheses results in a total of $-3 \times 49 = -147$ being added. To keep the function in balance, you must subtract -147 outside the parentheses.

$$y = -3(x^2 - 14x + 49) - 135 - (-147)$$

$$y = -3(x^2 - 14x + 49) - 135 + 147$$

$$y = -3(x - 7)^2 + 12$$

The vertex is at $(7, 12)$.

The next example involves fractions and a negative number in front of x^2 .

Example 11

Find the vertex for $y = -3x^2 + 8x + 1$.

Solution:

Place parentheses around the first two terms.

$$y = (-3x^2 + 8x) + 1$$

Factor -3 from the first two terms.

$$y = -3\left(x^2 - \frac{8}{3}x\right) + 1$$

Complete the square.

$$\left(\frac{1}{2} \times \frac{-8}{3}\right)^2 = \left(\frac{-4}{3}\right)^2$$

$$= \frac{16}{9}$$

$$y = -3\left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) + 1 - (-3)\left(\frac{16}{9}\right)$$

Adding $\frac{16}{9}$ inside the parentheses results in a total of $-3 \times \frac{16}{9} = -\frac{16}{3}$ being added. To keep things in balance, you must subtract $-\frac{16}{3}$ outside the parentheses. To subtract $-\frac{16}{3}$ is the same as adding positive $\frac{16}{3}$.

$$y = -3\left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) + 1 + \frac{16}{3}$$

$$y = -3\left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) + \frac{3}{3} + \frac{16}{3}$$

$$y = -3\left(x^2 - \frac{8}{3}x + \frac{16}{9}\right) + \frac{19}{3}$$

Factor and simplify.

$$y = -3\left(x - \frac{4}{3}\right)^2 + \frac{19}{3}$$

The vertex is at $\left(\frac{4}{3}, \frac{19}{3}\right)$.

It is time for you to try completing the square on your own. Do the exercises that follow.

1. What is a quadratic function?
2. Why is it easier to graph a quadratic function in the form $y = a(x-h)^2 + k$ than in the form $y = ax^2 + bx + c$?
3. What does the word quadratic imply?
4. Write the standard form for a quadratic function.
5. Write the completed square form for a quadratic function.
6. Express any two of the following functions in standard form and identify the values of a , b , and c .

- a. $y - 3x^2 = 2x$
- b. $y = -5 - 3x^2$
- c. $y - 5x + 3 = -4x^2$
- d. $y + 5 = 2x + 3x^2$

7. What is a perfect square trinomial?

8. If $x^2 + Mx + N$ is a perfect square trinomial, then what relationship must exist between M and N ?

9. What number must be added to make a perfect square trinomial? Do any four of the following.

- a. $x^2 + 12x +$
- b. $x^2 + 18x +$
- c. $x^2 + 5x +$
- d. $x^2 - 8x +$
- e. $x^2 - 3x +$
- f. $x^2 - x +$

10. Factor each of the following perfect square trinomials. Do any four of the following.

- a. $x^2 - 14x + 49$
- b. $x^2 - 2x + 1$
- c. $x^2 + 4x + 4$
- d. $x^2 + \frac{14}{3}x + \frac{49}{9}$
- e. $x^2 - x + \frac{1}{4}$
- f. $x^2 + \frac{3}{2}x + \frac{9}{16}$

11. Write each quadratic function in completed square form and identify the coordinates of the vertex. Do any two of the following.

- a. $y = x^2 - 8x$
- b. $y = x^2 + 18x$
- c. $y = x^2 - 3x$

12. Write each quadratic function in completed square form and identify the coordinates of the vertex.
Do any two of the following.

a. $y = x^2 - 10x + 10$

b. $y = x^2 - 3x + 7$

c. $y = x^2 + 2x + 5$

13. Write each quadratic function in completed square form and identify the coordinates of the vertex.
Do any two of the following.

a. $y = 2x^2 - 5x - 6$

b. $y = 3x^2 - 6x + 10$

c. $y = 5x^2 - 15x + 10$

14. Write each quadratic function in completed square form and identify the coordinates of the vertex.
Do any two of the following.

a. $y = -2x^2 + 16x - 29$

b. $y = -3x^2 - 6x + 4$

c. $y = -x^2 + 2x - 1$



For solutions to Activity 1, turn to Appendix A,
Topic 2.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.



Extra Help

The most difficult part of completing the square is deciding what numerical value you should add inside the parentheses and what numerical value you should subtract outside the parentheses. If you have difficulty with these concepts, then go over this section.

Example 12

Square one-half the value of each of the following numbers.

• 8

Solution:

$$\begin{aligned}\left(\frac{1}{2} \times 8\right)^2 &= \left(\frac{8}{2}\right)^2 \\ &= (4)^2 \\ &= 16\end{aligned}$$

• 7

Solution:

$$\begin{aligned}\left(\frac{1}{2} \times 7\right)^2 &= \left(\frac{7}{2}\right)^2 \\ &= \frac{49}{4}\end{aligned}$$

• -3

Solution:

$$\begin{aligned}\left(\frac{1}{2} \times -3\right)^2 &= \left(\frac{-3}{2}\right)^2 \\ &= \frac{9}{4}\end{aligned}$$

• -14

Solution:

$$\begin{aligned}\left(\frac{1}{2} \times -14\right)^2 &= \left(\frac{-14}{2}\right)^2 \\ &= (-7)^2 \\ &= 49\end{aligned}$$

• $\frac{3}{5}$

Solution:

$$\begin{aligned}\left(\frac{1}{2} \times \frac{3}{5}\right)^2 &= \left(\frac{3}{10}\right)^2 \\ &= \frac{9}{100}\end{aligned}$$

• -1

Solution:

$$\begin{aligned}\left(\frac{1}{2} \times -1\right)^2 &= \left(\frac{-1}{2}\right)^2 \\ &= \frac{1}{4}\end{aligned}$$



Now that you can square one-half of a given value, follow the examples to see how you would use this idea.

Example 13

Put $y = x^2 + 10x$ in completed square form.

Solution:

$$y = x^2 + 10x + ?$$

To complete the square, you must **add** a number at the end. The number that you add must be the square of one-half of the number in front of x .

$$\begin{aligned}\left(\frac{1}{2} \times 10\right)^2 &= \left(\frac{10}{2}\right)^2 \\ &= (5)^2 \\ &= 25\end{aligned}$$

If you add 25 at the end, then you must also subtract 25 to keep the function in balance.

$$y = x^2 + 10x + 25 - 25$$

The first three terms should be a perfect square trinomial. Put parentheses around them.

$$y = (x^2 + 10x + 25) - 25$$

Since the first three terms form a perfect square trinomial, you can factor them into a binomial that is squared.

$$y = (x + 5)^2 - 25$$

The completed square form is $y = (x + 5)^2 - 25$. The vertex is at $(-5, -25)$.



Example 14

Write $y = x^2 - 24x + 100$ in completed square form.

Solution:

Place parentheses around the first two terms.

$$y = (x^2 - 24x) + 100$$

Complete the square inside the parentheses.

$$\left(\frac{1}{2} \times -24\right)^2 = (-12)^2 \\ = 144$$

Add 144 inside the parentheses and subtract 144 outside the parentheses.

$$y = (x^2 - 24x + 144) + 100 - 144$$

Factor the perfect square trinomial inside the parentheses.

$$y = (x - 12)^2 - 44$$

The completed square form is $y = (x - 12)^2 - 44$. The vertex is at $(12, -44)$.

Example 15

Write $y = 3x^2 - 9x - 7$ in completed square form.

Solution:

Place parentheses around the first two terms.

$$y = (3x^2 - 9x) - 7$$

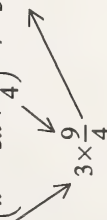
If there is a number in front of the x^2 , this number has to be factored from the two terms inside the parentheses.

$$y = 3(x^2 - 3x) - 7$$

Complete the square inside the parentheses.

$$\left(\frac{1}{2} \times -3\right)^2 = \left(\frac{-3}{2}\right)^2 \\ = \frac{9}{4}$$

Add $\frac{9}{4}$ inside the parentheses and subtract $3 \times \frac{9}{4}$ outside the parentheses.

$$y = 3\left(x^2 - 3x + \frac{9}{4}\right) - 7 - 3\left(\frac{9}{4}\right)$$


Factor the perfect square trinomial.

$$y = 3\left(x - \frac{3}{2}\right)^2 - 7 - \frac{27}{4}$$

$$y = 3\left(x - \frac{3}{2}\right)^2 - \frac{28}{4} - \frac{27}{4}$$

$$y = 3\left(x - \frac{3}{2}\right)^2 - \frac{55}{4}$$

The completed square form is $y = 3\left(x - \frac{3}{2}\right)^2 - \frac{55}{4}$. The vertex is at $\left(\frac{3}{2}, -\frac{55}{4}\right)$.

Example 16

Write $y = -4x^2 - 12x + 3$ in completed square form.

Solution:

Place parentheses around the first two terms.

$$y = (-4x^2 - 12x) + 3$$

Factor -4 from the first two terms.

$$y = -4\left(x^2 + 3x\right) + 3$$

Add $\frac{9}{4}$ inside the parentheses and subtract $(-4)\left(\frac{9}{4}\right)$ outside the parentheses.

$$y = -4\left(x^2 + 3x + \frac{9}{4}\right) + 3 - (-4)\left(\frac{9}{4}\right)$$

$\swarrow$ $\searrow$
 $-4 \times \frac{9}{4}$

$$y = -4\left(x + \frac{3}{2}\right)^2 + 3 + 4\left(\frac{9}{4}\right) \quad (\text{Recall } 3 - (-4) = 3 + 4.)$$

$$y = -4\left(x + \frac{3}{2}\right)^2 + 3 + 9$$

$$y = -4\left(x + \frac{3}{2}\right)^2 + 12$$

The completed square form is $y = -4\left(x + \frac{3}{2}\right)^2 + 12$. The vertex is at $\left(-\frac{3}{2}, 12\right)$.

Do the following exercises.

Write each of the following functions in completed square form and identify the coordinates of the vertex.

1. $y = x^2 + 9x$

2. $y = x^2 - 12x$

3. $y = x^2 + 6x - 3$

4. $y = x^2 - 8x + 7$

5. $y = 3x^2 - 15x + 1$

6. $y = 2x^2 + 5x - 3$

7. $-2x^2 - 8x + 1$

8. $y = -3x^2 + x - 2$



For solutions to Extra Help, turn to Appendix A, Topic 2.





Write the completed square form and the coordinates of the vertex for the following functions.

1. $y = 7x^2 - 2x + 1$

2. $y = \frac{1}{4}x^2 - \frac{2}{3}x + \frac{1}{5}$

3. $y = 1.72x^2 + 5.16x - 3.82$
(Use your calculator.)

4. $y = ax^2 + bx + c$



Topic 3 Characteristics of Quadratic Functions



Introduction

An object is projected upward with a velocity of 392 m/s . Can you calculate the height that this object will reach and the time taken to reach this height?

In this topic you will be expanding your knowledge of quadratic functions and learning how to solve problems such as the previous one.



What Lies Ahead

Throughout the topic you will learn to

1. identify and calculate the characteristics that define quadratic functions

Now that you know what to expect, turn the page to begin your study of characteristics of quadratic functions.



Exploring Topic 3

Activity 1



Identify and calculate the characteristics that define quadratic functions.

A flare is projected upward with a velocity of 60 m/s. The function that relates the height above ground to the time is $h = -5t^2 + 60t + 2$.

Can you use the previous function to determine the maximum height the flare reaches and the time required to reach this height?



By looking at the function, you can identify it as a quadratic function. Since it is a quadratic function, the graph of the function will be a parabola. The parabola will open downward because the number in front of the t^2 is negative.

To sketch the graph of $h = -5t^2 + 60t + 2$, you would complete the square.

$$h = (-5t^2 + 60t) + 2$$

$$h = -5(t^2 - 12t) + 2$$

$$h = -5(t^2 - 12t + 36) + 2 - (-5)(36)$$

$$h = -5(t - 6)^2 + 2 + 180$$

$$h = -5(t - 6)^2 + 182$$

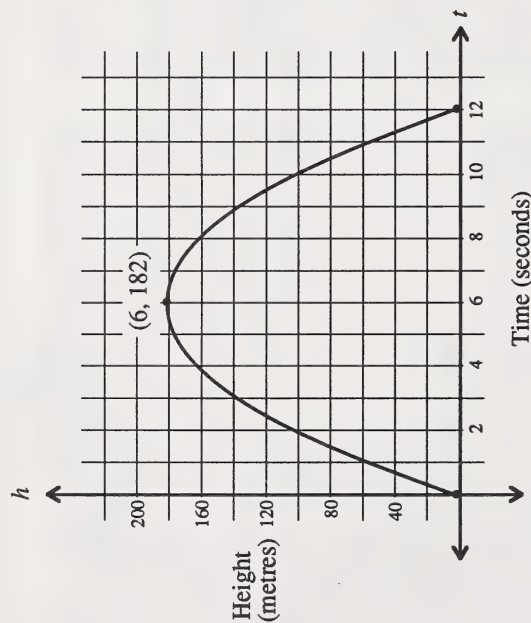
The vertex can be identified as $(6, 182)$. Now you are able to sketch the graph.

You can see that the maximum height occurs at the vertex. The coordinates of the vertex are at $(6, 182)$.

↓ ↓
time height

In this example, x and y have been replaced by t and h , where t stands for time and h stands for height above ground.

The maximum height reached by the flare is 182 m and it takes 6 s for the flare to reach this height.



The previous example demonstrates that if the coordinates of the vertex of a parabola are known, the maximum value of the function is equal to the **second coordinate** of the vertex.

In dealing with maximum and minimum values of a function, two cases must be considered.

If the vertex is at (h, k) , the maximum value is k since k is the second coordinate.

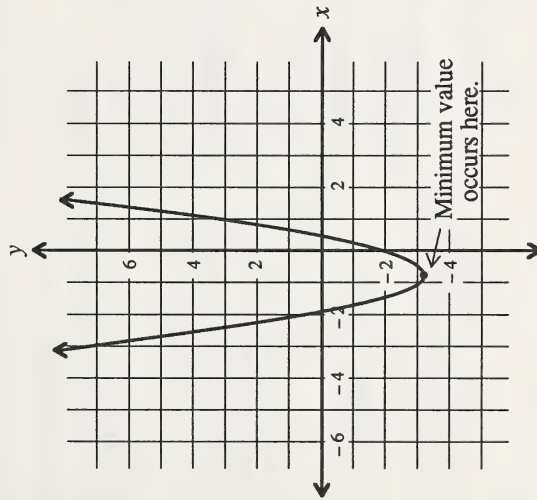
Case 1



$$y = ax^2 + bx + c, \text{ where } a \text{ is positive}$$

$$y = 2x^2 + 3x - 2$$

When a is positive, the parabola opens upward. Since the parabola opens upward, a **minimum** value occurs at the vertex.



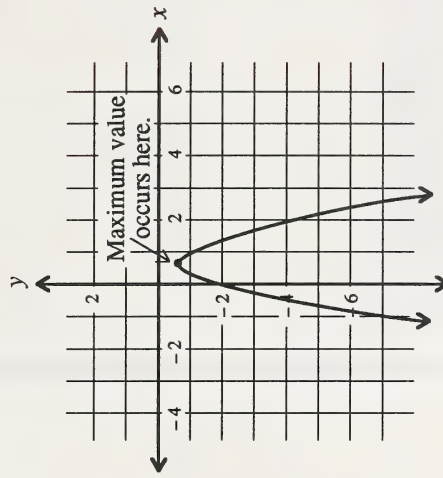
Case 2



$$y = ax^2 + bx + c, \text{ where } a \text{ is negative}$$

$$y = -2x^2 + 3x - 2$$

When a is negative, the parabola opens downward. Since the parabola opens downward, a **maximum** value occurs at the vertex.



You can tell by looking at the equation of the function whether a maximum or a minimum value will occur. Follow the next example.

Example 1

For the following quadratic functions, identify the value of a and tell whether a maximum or minimum occurs.

- $y = -3x^2 + 5x - 6$

Solution:

$$y = -3x^2 + 5x - 6$$

$$a = -3$$

The function has a maximum.

- $y + 2x^2 = 3x + 4$

Solution:

$$y + 2x^2 = 3x + 4$$

$$y = -2x^2 + 3x + 4$$

$$a = -2$$

The function has a maximum.

- $y = 5(x - 3)^2 + 4$

Solution:

$$y = 5(x - 3)^2 + 4$$

$$a = 5$$

The function has a minimum.

- $\frac{2}{3}x^2 + 5x = y$

Solution:

$$y = \frac{2}{3}x^2 + 5x$$

$$a = \frac{2}{3}$$

The function has a minimum.

- $y = -(x + 2)^2 - 3$

Solution:

$$y = -(x + 2)^2 - 3$$

$$a = -1$$

The function has a maximum.

- $y - 5x^2 = 3$

Solution:

$$y - 5x^2 = 3$$

$$y = 5x^2 + 3$$

$$a = 5$$

The function has a minimum.

To identify the value of a , always put the function in either standard form or completed square form.

$$y = ax^2 + bx + c \text{ or}$$

$$y = a(x - h)^2 + k$$

Now you can tell whether a maximum or minimum value exists. It is time for you to find the value of the maximum or minimum.



The maximum or minimum value of a quadratic function occurs at the vertex. The maximum or minimum value is equal to the numerical value of the second coordinate of the vertex. If the vertex has coordinates of (h, k) , then the maximum or minimum value of that function is k .

The maximum or minimum value of a quadratic function is the value of the **dependent variable** at the vertex. The dependent variable is always written as the second coordinate of an ordered pair.

$y = 2x^2 + 5x - 6$ $h = -9t^2 + 108t + 2$
 dependent variable dependent variable
 independent variable independent variable

For $y = 2x^2 + 5x - 6$, the minimum value of the function is the y -value at the vertex.

For $h = -9t^2 + 108t + 2$, the maximum value of the function is the h -value at the vertex.

Follow the next examples to learn more about maximum and minimum values.

Example 2

For the quadratic function $y = 2(x - 5)^2 - 4$, answer the following:

- Identify the value of a and state whether a maximum or minimum value occurs.

Solution:

Since $a = 2$, a minimum value occurs.

- Calculate the maximum or minimum value of the function.

Solution:

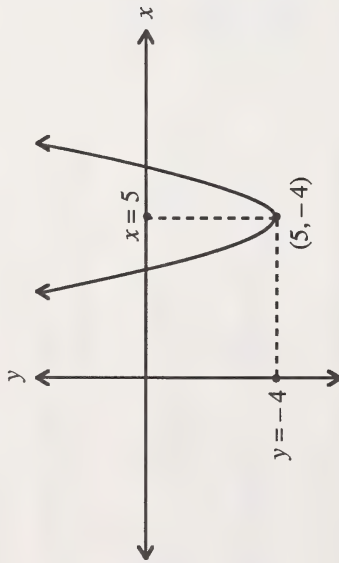
The vertex is at $(5, -4)$; therefore, the minimum value is -4 since $y = -4$. The minimum occurs when $x = 5$.

The maximum value for a function is the biggest possible value for the dependent variable. The minimum value is the smallest possible value for the dependent variable.

Often, variables other than x and y are used in a function.

- Sketch the graph of the function showing where the maximum or minimum value is.

Solution:



- Calculate the maximum or minimum value of the function.

Solution:

To find the vertex, you must write the function in completed square form.

$$y = -3(x^2 + 8x) + 3$$

$$y = -3(x^2 + 8x + 16) + 3 - (-3)(16)$$

$$y = -3(x + 4)^2 + 3 + 48$$

$$y = -3(x + 4)^2 + 51$$

The vertex is at $(-4, 51)$; therefore, the maximum value is 51 since $y = 51$. This maximum occurs when $x = -4$.

Example 3

For the quadratic function $y = -3x^2 - 24x + 3$, answer the following:

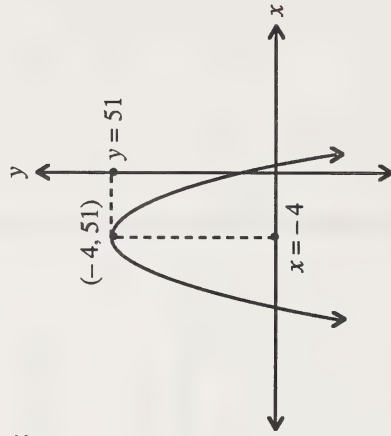
- Identify the value of a and state whether a maximum or minimum value occurs.

Solution:

Since $a = -3$, a maximum value occurs.

- Sketch the graph of the function showing where the maximum or minimum value is.

Solution:



Finding the maximum or minimum value of a quadratic function simply involves finding the coordinates of the vertex and then identifying the second coordinate of the vertex as either the maximum or minimum value.



The **axis of symmetry** of a parabola is a line that divides the parabola into two parts which are mirror images of each other.

The axis of symmetry must

- be a vertical line
- pass through the vertex



If the coordinates of the vertex of a parabola are (h, k) , then the axis of symmetry for that parabola is defined by the equation $x = h$.

Look at the following example.

Example 4

For the quadratic function $y = -2(x - 3)^2 + 7$, answer the following:

- Identify the vertex.

Solution:

The vertex is at $(3, 7)$.

- State the location of the axis of symmetry.

Solution:

The axis of symmetry is a vertical line passing through $(3, 7)$.

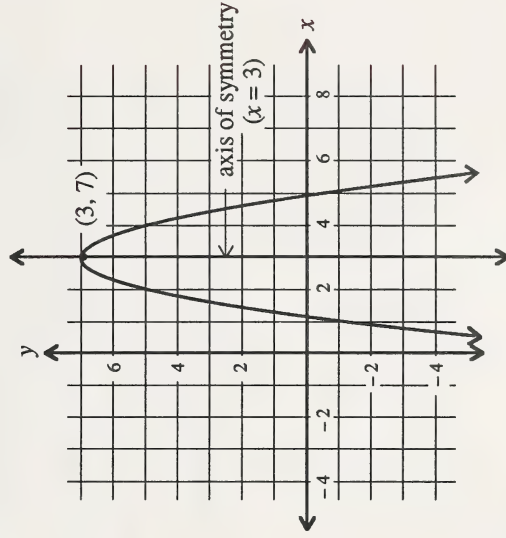
- Write the equation for the axis of symmetry.

Solution:

The axis of symmetry is defined by the equation $x = 3$.

- Graph the function showing the axis of symmetry.

Solution:



Example 5

For the quadratic function $y = 3x^2 - 6x + 5$, answer the following:

- Identify the vertex.

Solution:

$$y = 3(x^2 - 2x) + 5$$

$$y = 3(x^2 - 2x + 1) + 5 - 3$$

$$y = 3(x - 1)^2 + 2$$

The vertex is at $(1, 2)$.

- State the location of the axis of symmetry.

Solution:

The axis of symmetry is a vertical line passing through $(1, 2)$.

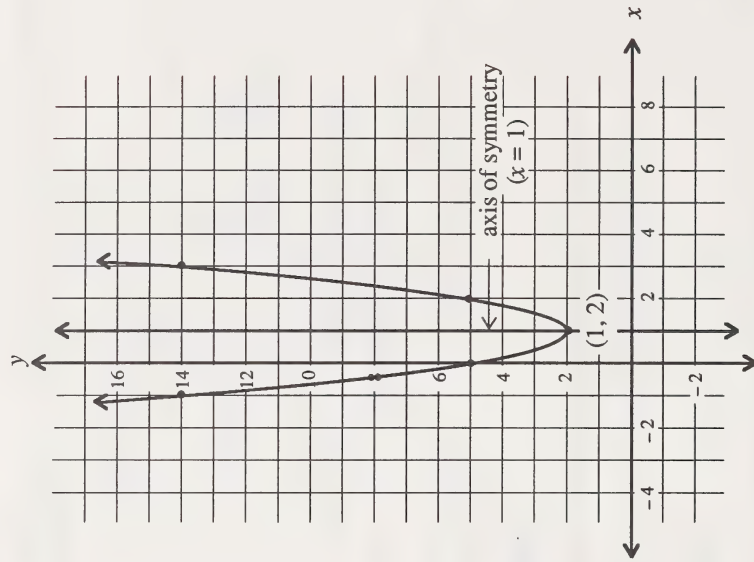
- Write the equation for the axis of symmetry.

Solution:

The equation for the axis of symmetry is $x = 1$.

- Graph the function showing the axis of symmetry.

Solution:



The following examples show how the vertex, the axis of symmetry, and the maximum or minimum value of a quadratic equation are related. Study the examples carefully.

Example 6

For the quadratic function $z = -5(m+4)^2 + 8$, identify the following:

- the coordinates of the vertex

Solution:

The vertex is at $(-4, 8)$.

- the axis of symmetry

Solution:

The axis of symmetry is $m = -4$.

- the maximum or minimum value of the function

Solution:

The maximum value is 8 since $z = 8$.

Quadratic Function	Coordinates of the Vertex
$y = 2(x-3)^2 + 5$	$(3, 5)$ The equation for the axis of symmetry is $x = 3$. The minimum value of the function is 5 since $y = 5$.
$y = a(x-h)^2 + k$	(h, k) The equation for the axis of symmetry is $x = h$. The maximum or minimum value of the function is k since $y = k$.
$y = -7(x+\sqrt{2})^2 + \pi$	$(-\sqrt{2}, \pi)$ The equation for the axis of symmetry is $x = -\sqrt{2}$. The maximum value of the function is π since $y = \pi$.

How do you determine the axis of symmetry and the maximum or minimum value if variables other than x and y are used? Look at the next example.

Example 7

For the quadratic function $p = 2r^2 + 5r - 6$, identify the following:

- the coordinates of the vertex

Solution:

Put into completed square form in order to identify the coordinates of the vertex.

$$p = 2\left(r^2 + \frac{5}{2}r\right) - 6$$

$$p = 2\left(r^2 + \frac{5}{2}r + \frac{25}{16}\right) - 6 - 2\left(\frac{25}{16}\right)$$

$$p = 2\left(r + \frac{5}{4}\right)^2 - 6 - \frac{25}{8}$$

$$p = 2\left(r + \frac{5}{4}\right)^2 - \frac{73}{8}$$

The vertex is at $\left(-\frac{5}{4}, -\frac{73}{8}\right)$.

- the axis of symmetry

Solution:

The axis of symmetry is $r = -\frac{5}{4}$.

- the maximum or minimum value of the function

Solution:

The minimum value is $-\frac{73}{8}$ since $p = -\frac{73}{8}$.

In Unit 3 you were shown how to find the domain and the range for various functions and relations. You will now apply what you learned to find the domain and the range of quadratic functions. Recall the following:

- The domain refers specifically to the set of possible values for the independent variable.
- The range refers specifically to the set of possible values for the dependent variable.

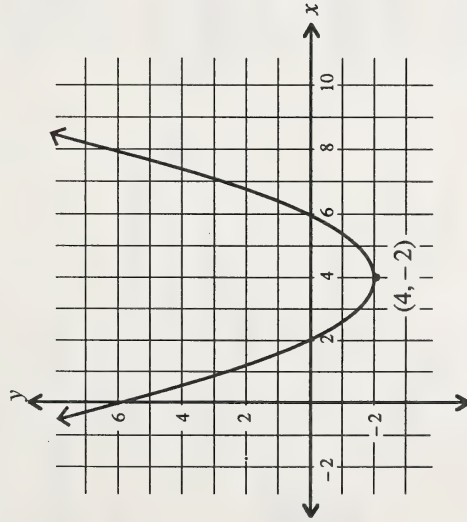
For the quadratic function $y = ax^2 + bx + c$ or $y = a(x - h)^2 + k$, the following is true:

- The domain refers to the set of x -values (independent variable).
- The range refers to the set of y -values (dependent variable).

Domain and range can be best explained by a graphic illustration. Look at the next example.

Example 8

Determine the domain and the range for each parabola.



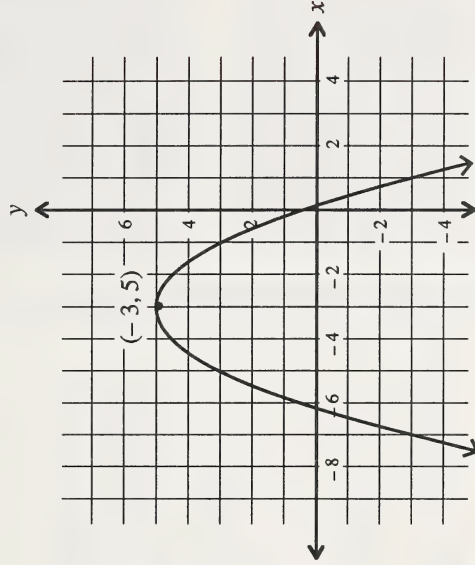
Solution:

Domain

The arrows indicate an infinite continuation of the parabola. Since the parabola continues in the upward direction with no end, every point on the x -axis must have a corresponding point on the parabola with the same x -value. This means that the domain is the set of real numbers. This is written as $x \in R$.

Range

The parabola opens upward; therefore, the parabola has a minimum value of -2 since $y = 2$ at the vertex. Since $y = -2$ is the minimum value of y , then all other values of y must be greater than -2 . This means that the range must be $y \geq -2$.



Solution:

Domain

The arrows indicate an infinite continuation of the parabola. Since the parabola continues in the downward direction with no end, every point on the x -axis must have a corresponding point on the parabola with the same x -value. This means that the domain is the set of real numbers. This is written as $x \in R$.

Range

The parabola opens downward; therefore, the parabola has a maximum value of 5 since $y = 5$ at the vertex. Since $y = 5$ is the maximum value of y , then all other values of y must be less than 5. This means that the range must be $y \leq 5$.

In Example 8, you can see that the range depends upon the maximum or minimum value of the function.

The domain and range for the quadratic function

$$y = a(x - h)^2 + k, \text{ with the}$$

vertex at (h, k) , is as follows:

- The domain is always $x \in R$.
- The range is $y \geq k$ if the parabola has a minimum value ($a > 0$), and $y \leq k$ if the parabola has a maximum value ($a < 0$).



Example 8 showed you how to find the domain and the range from the graph of a quadratic function. The next example shows you how to find the domain and the range from the equation of a quadratic function.

Example 9

For the quadratic function $y = -\frac{1}{2}\left(x + \frac{2}{3}\right)^2 - 3$, identify the following:

- the coordinates of the vertex

Solution:

The vertex is at $\left(-\frac{2}{3}, -3\right)$.

- the domain

Solution:

The domain is $x \in R$.

- the range

Solution:

The parabola opens downward since $a = -\frac{1}{2}$ and it has a maximum value of -3 . Therefore, the range is $y \leq -3$.

$$y \geq -2$$

This sign means that y is greater than or equal to -2 .

$$y \leq 5$$

This sign means that y is less than or equal to 5.

Example 10

For the quadratic function $y = \frac{2}{3}x^2 + \frac{2}{9}x - \frac{5}{6}$, identify the following:

- the coordinates of the vertex

Solution:

$$y = \frac{2}{3}x^2 + \frac{2}{9}x - \frac{5}{6}$$

$$y = \left(\frac{2}{3}x^2 + \frac{2}{9}x\right) - \frac{5}{6}$$

$$y = \frac{2}{3}\left(x^2 + \frac{1}{3}x\right) - \frac{5}{6}$$

$$y = \frac{2}{3}\left(x^2 + \frac{1}{3}x + \frac{1}{36}\right) - \frac{5}{6} - \frac{2}{3}\left(\frac{1}{36}\right)$$

$$y = \frac{2}{3}\left(x + \frac{1}{6}\right)^2 - \frac{5}{6} - \frac{1}{54}$$

$$y = \frac{2}{3}\left(x + \frac{1}{6}\right)^2 - \frac{23}{27}$$

The vertex is at $\left(-\frac{1}{6}, -\frac{23}{27}\right)$.

- the domain

Solution:

The domain is $x \in \mathbb{R}$.

- the range

Solution:

The parabola opens upward since $a = \frac{2}{3}$ and it has a minimum value of $-\frac{23}{27}$. Therefore, the range is $y \geq -\frac{23}{27}$.

From the previous examples you can see how important it is that you are able to calculate the coordinates of the vertex. From the coordinates of the vertex you can identify the following:

- the maximum or minimum value
- the axis of symmetry
- the range of the function

In **Mathematics 13**, you learned how to find the y-intercept for a linear function.

The y-intercept for the graph of a function is the point where the graph crosses the y-axis.

The next example shows you how to identify the y-intercept from a graph.

To factor $\frac{2}{3}x^2 + \frac{2}{9}x$, divide each term by $\frac{2}{3}$.

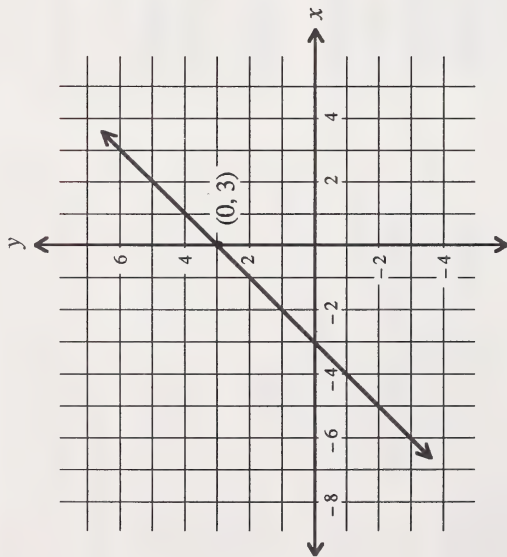
$$\begin{aligned} \frac{\frac{2}{3}x^2}{\frac{2}{3}} + \frac{\frac{2}{9}x}{\frac{2}{3}} &= x^2 + \frac{2}{9}x\left(\frac{3}{2}\right) \\ &= x^2 + \frac{1}{3}x \end{aligned}$$

Do not forget to put $\frac{2}{3}$ in front of the parentheses.

$$\begin{aligned} -\frac{5}{6} - \frac{1}{54} &= -\frac{45}{54} - \frac{1}{54} \\ &= -\frac{46}{54} \\ &= -\frac{23}{27} \end{aligned}$$

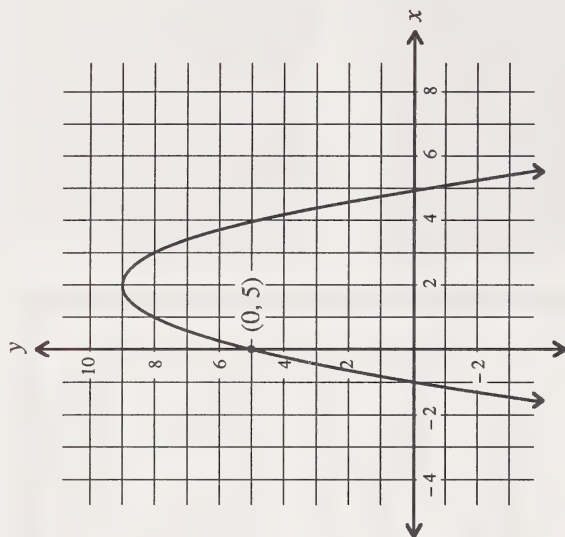
Example 11

Find the y-intercept for the following functions.



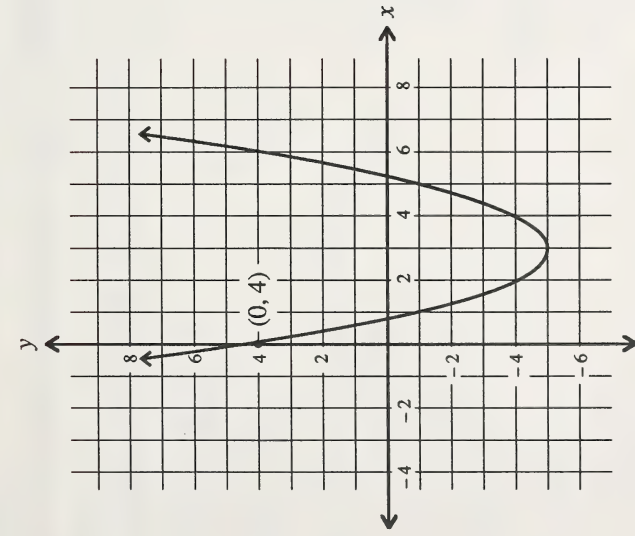
Solution:

The graph crosses the y-axis at $(0, 3)$. The y-intercept is 3.



Solution:

The graph crosses the y-axis at $(0, 5)$. The y-intercept is 5.



Solution:

The graph crosses the y -axis at $(0, 4)$. The y -intercept is 4.

From the previous example you can see that the x -value is zero at each y -intercept.

The y -intercepts for the three graphs are located at $(0, 3)$, $(0, 5)$, and $(0, 4)$.

Use the previous idea to find the y -intercept of a function when the equation is given. Look at the next example to see how this is done.

Example 12

Find the y -intercept for each function that follows.

- $y = 3x^2 + 2x + 5$

Solution:

$$y = 3x^2 + 2x + 5$$



To find the y -intercept from the equation of a function, replace every x by 0 (zero) and then solve for y .

Let $x = 0$.

$$y = 3(0)^2 + 2(0) + 5$$

$$y = 0 + 0 + 5$$

$$y = 5$$

The y -intercept is 5.

- $y = -2(x+3)^2 - 4$

Solution:

$$y = -2(x+3)^2 - 4$$

$$\text{Let } x = 0.$$

$$y = -2(0+3)^2 - 4$$

$$y = -2(3)^2 - 4$$

$$y = -2(9) - 4$$

$$y = -18 - 4$$

$$y = -22$$

The y-intercept is -22 .

- $y = -3x^2 - 2x - 4$

Solution:

$$y = -3x^2 - 2x - 4$$

$$\text{Let } x = 0.$$

$$y = -3(0)^2 - 2(0) - 4$$

$$y = -3(0) - 0 - 4$$

$$y = 0 - 0 - 4$$

$$y = -4$$

The y-intercept is -4 .

It is time for you to put together all that you have learned in this topic. Look at the following example.

Example 13

For the quadratic function $y = 3(x+2)^2 - 3$, do the following:

- Find the coordinates of the vertex.

Solution:

$$y = 3(x+2)^2 - 3$$

The vertex is at $(-2, -3)$.

- Find the maximum or minimum value.

Solution:

The graph opens upward; therefore, it has a minimum value of -3 since $y = -3$.

- State the axis of symmetry.

Solution:

The equation for the axis of symmetry is $x = -2$.

- State the domain.

Solution:

The domain is $x \in \mathbb{R}$.

- State the range.

Solution:

The range is $y \geq -3$.

- Find the y-intercept.

Solution:

The y-intercept is found by letting $x = 0$.

$$y = 3(0 + 2)^2 - 3$$

$$y = 3(2)^2 - 3$$

$$y = 3(4) - 3$$

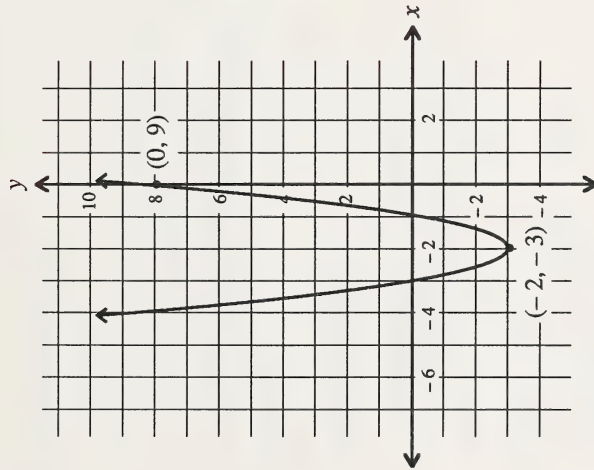
$$y = 12 - 3$$

$$y = 9$$

The y-intercept is 9.

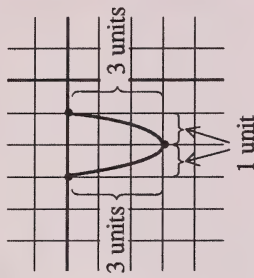
- Sketch the graph of the function.

Solution:



Recall how you may approximately sketch the shape of a parabola if the value of a is known.

$$a = 3$$



If $a = 3$, go one unit to the left and one unit to the right of the vertex, and then go three units up on each side.

Example 14

For the quadratic function $y = -2x^2 - 10x - 8$, do the following:

- Find the coordinates of the vertex.

Solution:

Put the equation into completed square form to find the vertex.

$$y = -2\left(x^2 + 5x\right) - 8$$

$$y = -2\left(x^2 + 5x + \frac{25}{4}\right) - 8 - (-2)\left(\frac{25}{4}\right)$$

$$y = -2\left(x + \frac{5}{2}\right)^2 - 8 + \frac{25}{2}$$

$$y = -2\left(x + \frac{5}{2}\right)^2 - \frac{16}{2} + \frac{25}{2}$$

$$y = -2\left(x + \frac{5}{2}\right)^2 + \frac{9}{2}$$

The vertex is at $\left(-\frac{5}{2}, \frac{9}{2}\right)$.

- Find the maximum or minimum value.

Solution:

The graph opens downward; therefore, it has a maximum value of $\frac{9}{2}$ since $y = \frac{9}{2}$.

- State the axis of symmetry.

Solution:

The equation for the axis of symmetry is

$$x = -\frac{5}{2}.$$

- State the domain.

Solution:

The domain is $x \in R$.

- State the range.

Solution:

The range is $y \leq \frac{9}{2}$.

- Find the y-intercept.

Solution:

To find the y-intercept, let $x = 0$.

$$y = -2(0)^2 - 10(0) - 8$$

$$y = 0 - 0 - 8$$

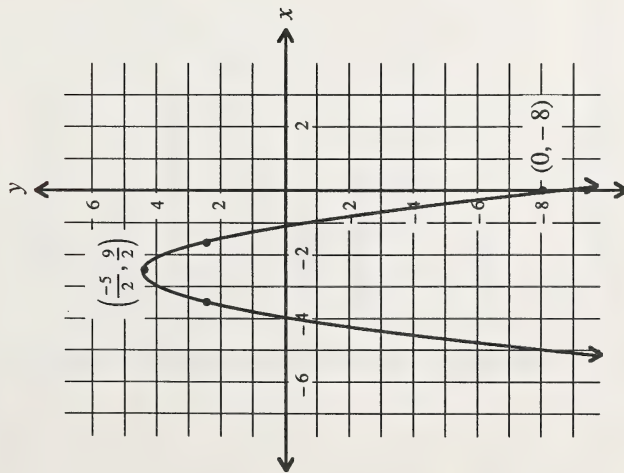
$$y = -8$$

The y-intercept is -8 .

If the equation is in the form
 $y = ax^2 + bx + c$, the y-intercept
will always be c .

- Sketch the graph of the function.

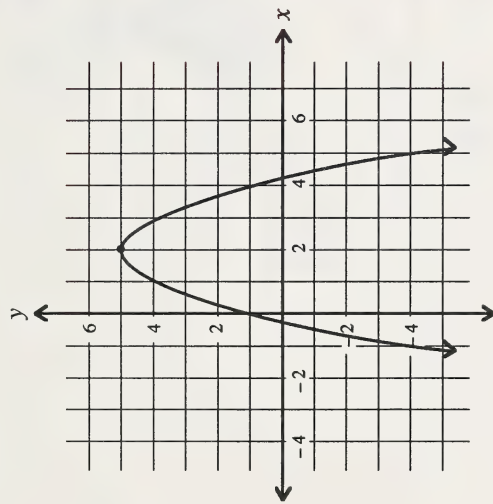
Solution:



The next example shows you how to obtain information from a graph.

Example 15

For the next graph, do the following:

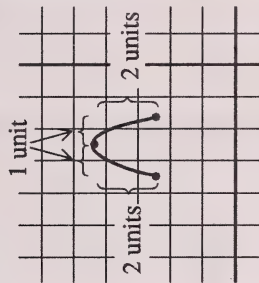


- Write the coordinates of the vertex.

Solution:

The vertex is at $(2, 5)$.

$$a = -2$$



If a is -2 , go one unit to the left and right of the vertex, and then go two units down on both sides.

- Write the maximum or minimum value.

Solution:

The graph has a maximum value of 5 since $y = 5$.

- State the equation for the axis of symmetry.

Solution:

The axis of symmetry is $x = 2$.

- State the domain.

Solution:

The domain is $x \in R$.

- State the range.

Solution:

The range is $y \leq 5$.

- State the y-intercept.

Solution:

The y-intercept is 1.

- Write the equation of the function in completed square form and in standard form.

Solution:

The equation of the function in completed square form is as follows:

$$a = -1$$

$$y = -1(x - 2)^2 + 5$$

The equation of the function in standard form is as follows:

$$y = -1(x^2 - 4x + 4) + 5$$

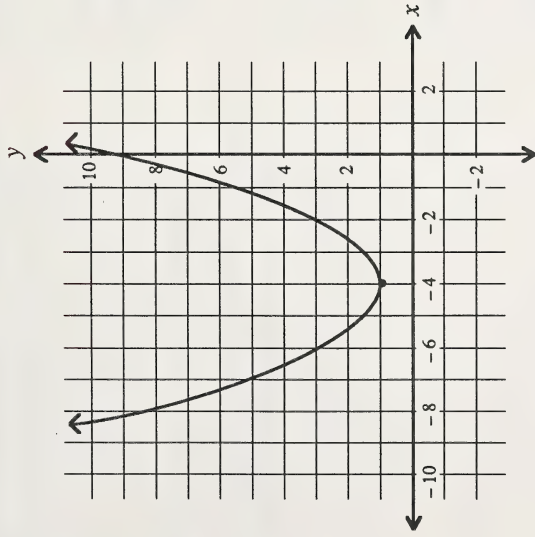
$$y = -x^2 + 4x - 4 + 5$$

$$y = -x^2 + 4x + 1$$



Example 16

For the next graph, do the following:



- Write the coordinates of the vertex.

Solution:

The vertex is at $(-4, 1)$.

- Write the maximum or minimum value.

Solution:

The graph has a minimum value of 1 since $y = 1$.

- State the axis of symmetry.

Solution:

The axis of symmetry is $x = -4$.

- State the domain.

Solution:

The domain is $x \in \mathbb{R}$.

- State the range.

Solution:

The range is $y \geq 1$.

- State the y-intercept.

Solution:

The y-intercept is 9.

- Write the equation of the function in completed square form and in standard form.

Solution:

The equation of the function in completed square form is as follows:

$$a = \frac{1}{2}$$

$$y = \frac{1}{2}(x+4)^2 + 1$$

The equation of the function in standard form is as follows:

$$y = \frac{1}{2}(x^2 + 8x + 16) + 1$$

$$y = \frac{1}{2}x^2 + 4x + 8 + 1$$

$$y = \frac{1}{2}x^2 + 4x + 9$$

The remainder of this topic deals with practical applications of what you have learned. Many problems in life deal with finding the maximum or minimum value in a particular situation. Many businesses use quadratic functions to maximize their profits or minimize their expenses. Scientists quite often use quadratic functions to determine the maximum height an object will attain. The following examples should give you some idea as to how to solve problems dealing with maximum and minimum values.

Example 17

A missile is projected upward with an initial velocity of 1500 m/s. The function that relates the height above ground (in metres) to the time (in seconds) is given by $h = -5t^2 + 1500t$. Neglecting air resistance, find the following:

- the maximum height the missile reaches

Solution:

$$h = -5t^2 + 1500t \quad \text{(Put into completed square form.)}$$

$$h = -5(t^2 - 300t) \quad \text{)$$

$$h = -5(t^2 - 300t + 22\,500) - (-5)(22\,500)$$

$$h = -5(t - 150)^2 + 5(22\,500)$$

$$h = -5(t - 150)^2 + 112\,500 \quad \text{(completed square form)}$$

This value is the time required to reach maximum height.

This value is the maximum height.

The maximum height reached is 112 500 m or 112.5 km.

$$h = -5t^2 + 1500t$$



$a = -5$; therefore, a maximum will occur.

$$\left(\frac{1}{2} \times -300\right)^2 = (-150)^2 = 22\,500$$

$$\sqrt{22\,500} = 150$$

- the time needed to reach the maximum height

Solution:

The time needed to reach the maximum height is 150 s.

- the total time the missile is in the air

Solution:

The total time the missile is in the air is $150 \text{ s} \times 2 = 300 \text{ s}$.

Quite often, the equation for a function will not be given. It will be up to you to write your own equation. The next example shows you how this is done.

Example 18

Find two numbers whose sum is 30 and whose product is a maximum. What is the maximum product?

Solution:

Since both numbers are unknown, you must first define them in terms of a variable.

Let n be one of the numbers. Therefore, $30 - n$ must be the other number.

$$p = n(30 - n)$$

This function shows the relationship between the two numbers and their product.

$$p = 30n - n^2$$

This function is a quadratic function.

$$p = -n^2 + 30n \quad (\text{Put into standard form.})$$

$$p = -1(n^2 - 30n) \quad (\text{Put into completed square form.})$$

$$p = -1(n^2 - 30n + 225) - (-1)(225)$$

$$p = -1(n - 15)^2 + 225 \quad (\text{completed square form})$$

This value is the value of n at which a maximum product occurs.

↑

↑

This value is the value of n at which a maximum product occurs.

The maximum occurs when $n = 15$.

One of the numbers is 15 and the other number is 15 since $30 - 15 = 15$. The maximum product is 225.

$$p = -n^2 + 30n$$



$a = -1$; therefore, a maximum will occur.

One number was defined as n .
The other number was defined as $30 - n$.

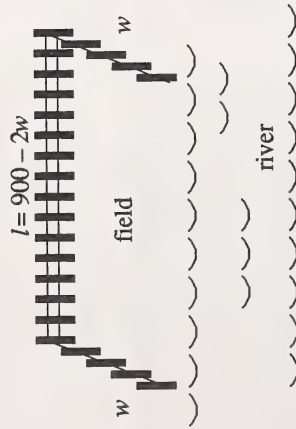
The use of a diagram may help you to solve some maximum or minimum problems. Look at the next example.

Example 19

A farmer has 900 m of fencing with which he wants to enclose a rectangular field bordered on one side by a river. If he does not have to build a fence along the river side, find the length and width of this field in order for a maximum area to be enclosed.

Solution:

Draw a diagram representing the situation.



Since the length and width are the unknown quantities, you must define them in terms of a variable.

Let w equal the width of the field.

The length is equal to $900 - 2w$.

The area of any rectangular region is found by multiplying the length by the width.

In this case,

$$A = (900 - 2w)w$$

$$A = 900w - 2w^2$$

$$A = -2w^2 + 900w$$

(This function is quadratic and has a maximum value ($a = -2$).)

$$A = -2(w^2 - 450w) \quad \text{(Put into completed square form.)}$$

$$A = -2(w^2 - 450w + 50\,625) - (-2)(50\,625)$$

$$A = -2(w - 225)^2 + 101\,250 \quad \text{(completed square form)}$$

$\uparrow$ This value is the width at which the maximum area occurs.
 $\uparrow$ This value is the maximum area.

The width of the field is 225 m.

The length of the field = $900 - 2(225)$

$$= 900 - 450$$

$$= 450 \text{ m}$$

The maximum area enclosed is $101\,250 \text{ m}^2$.

Example 20

An apple orchard contains twenty trees per hectare. Each tree yields approximately 400 apples. It has been established that for each additional tree planted per hectare, the yield per tree will be reduced by 10 apples. How many additional trees should be planted per hectare in order to obtain a maximum yield? What is the maximum yield?

Solution:

Let x equal the number of additional trees planted per hectare. The number of trees per hectare will be $(20 + x)$.

If x additional trees are planted per hectare, then the yield per tree will be $(400 - 10x)$.

You can see that the yield per hectare (y) must be the product of the number of trees per hectare $(20 + x)$ and the yield per tree $(400 - 10x)$.

$$\begin{aligned}
 y &= (20 + x)(400 - 10x) && \text{(Multiply.)} \\
 y &= 8000 - 200x + 400x - 10x^2 && \text{(Simplify and put into standard form.)} \\
 y &= -10x^2 + 200x + 8000 \\
 y &= -10(x^2 - 20x) + 8000 && \text{(Put into completed square form.)}
 \end{aligned}$$

$$\begin{aligned}
 y &= -10(x^2 - 20x + 100) + 8000 - (-10)(100) \\
 y &= -10(x - 10)^2 + 8000 + 1000 \\
 y &= -10(x - 10)^2 + 9000 && \text{(completed square form)}
 \end{aligned}$$

$\uparrow$ $\uparrow$
 This value is the number of additional trees planted per hectare. This value is the maximum yield.

A maximum yield of 9000 apples per hectare can be obtained if ten additional trees are planted per hectare.

$$\begin{aligned}
 \text{width} &= w \\
 \text{length} &= 900 - 2w
 \end{aligned}$$

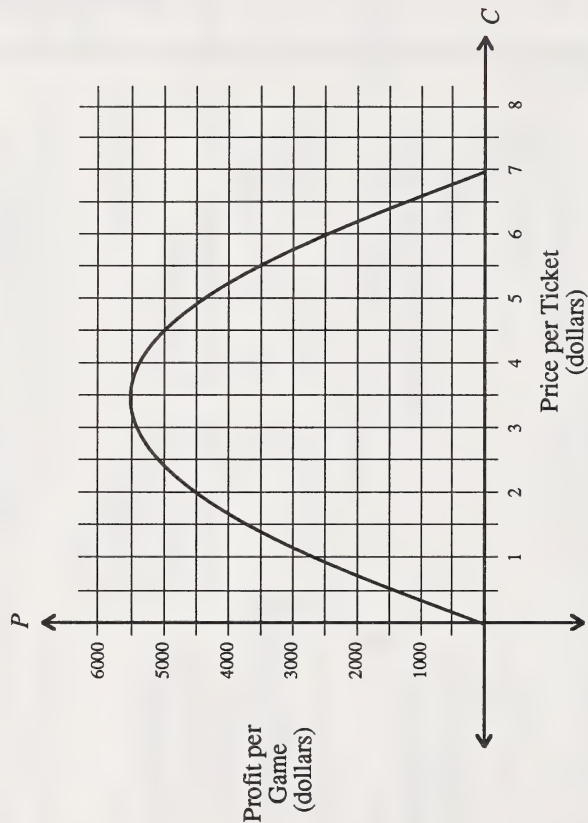
For each additional tree planted, the yield per tree is reduced by ten apples.

$$400 - 10x$$

The next example involves obtaining information from a graph.

Example 21

The next graph shows the relationship between the profit made per cricket game and the price charged per admission ticket. Using the graph, determine the following:



- the price per ticket such that a maximum profit occurs

Solution:

For maximum profit, the price per ticket should be \$3.50.

- the maximum profit

Solution:

The maximum profit is \$5500.

- the equation of the function that describes this graph if the value of a is approximately -450

Solution:

The vertex has coordinates of $(3.5, 5500)$ and $a = -450$.
Therefore, $P = -450(c - 3.5)^2 + 5500$ is the equation.

Try the following questions. Graph paper is provided in **Appendix B**.

1. By looking at the equation for a quadratic function, how can you tell whether a maximum or a minimum value occurs?
2. For each quadratic function that follows, determine the value of a and state whether a maximum or a minimum occurs. Do any two.

a. $y = 3x^2 + 5x - 2$

b. $y = 5 - 3x^2 + 7x$

c. $y = -5(x + 2)^2 - 7$

d. $y - 5(x - 2)^2 = 3$

3. What exactly is the minimum or maximum value of a quadratic function?

4. The graph of a quadratic function has its vertex at $(5, 6)$. Write a statement explaining how the values 5 and 6 are related.

5. For the function $y = 2(x + 3)^2 + 8$, what value of x will give a minimum value?

6. For each function that follows, state the minimum or maximum value. Do any two.

a. $y = -2(x + 7)^2 - 3$

b. $y = 4(x - \sqrt{2})^2 + 4$

c. $y = x^2 - 7$

7. For each function that follows, state the minimum or maximum value. Do any two.

a. $y = 2x^2 - 8x + 7$

b. $y = -3x^2 - 6x + 2$

c. $y = -x^2 + 7x$

8. Define the axis of symmetry of a parabola and explain where it can be found.

9. For each function that follows, write the equation for the axis of symmetry. Do any two.

a. $y = 2(x - 3)^2 + 6$

b. $y = -3(x + 5)^2 + 2$

c. $y = 2x^2 + 6$

10. Sketch the graph of $y = 2(x - 1)^2 + 3$ and show the location of the axis of symmetry.

11. State the domain and the range for the following quadratic functions. Do any two.
- $y = -5(x+3)^2 - 2$
 - $y = 2(x-5)^2 + 6$
 - $y = 5x^2 - 7$
12. What two things can you identify from the coordinates of the vertex of a parabola?
13. What is the y-intercept of a function?
14. Find the y-intercept for each function that follows. Do any two questions.

- $y = 5x^2 + 6x$
- $y = -2(x+3)^2 - 5$
- $y = 2x^2 + 6x - 5$

15. For any two of the quadratic functions, do the following:

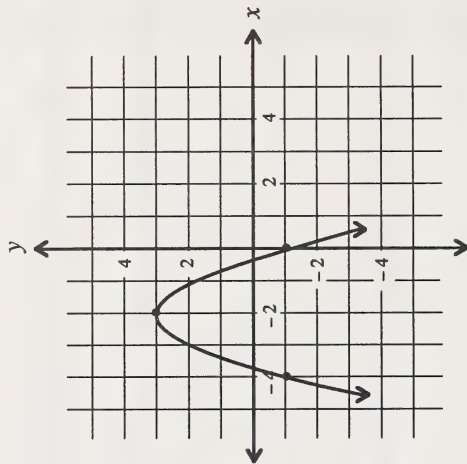
- Find the coordinates of the vertex.
- State the maximum or minimum value.
- State the axis of symmetry.
- State the domain.
- State the range.
- Find the y-intercept.
- Sketch the graph of the function.

- $y = -2(x+3)^2 - 1$
- $y = 2x^2 + 6x - 3$
- $y = 3(x-1)^2 + 2$
- $y = -3x^2 + 6x - 2$

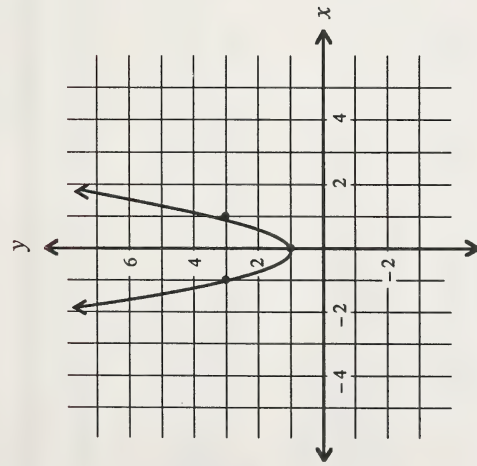
16. For any two of the parabolas, do the following:

- Write the coordinates of the vertex.
- State the maximum or minimum value.
- State the equation for the axis of symmetry.
- State the domain.
- State the range.
- Find the y-intercept.
- Write the equation of the function.

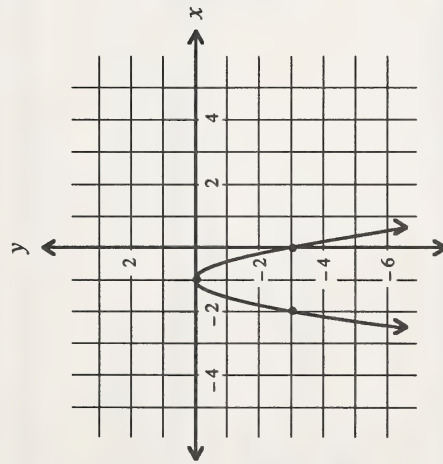
a.



b.



c.



17. The trajectory of a baseball is described by the function $h = -5t^2 + 60t$, where h is the height above the ground in metres and t is the time in seconds. Find the following:

- the maximum height reached by the baseball
- the time required to reach the maximum height
- the total time the baseball is airborne

18. What is the maximum rectangular area that can be enclosed with 1000 m of fencing? What are the dimensions of this rectangular area?

19. The sum of two numbers is 48 and the sum of their squares is a minimum. Find the two numbers.



For solutions to Activity 1, turn to Appendix A, Topic 3.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.



Extra Help

If the equation for a quadratic function is in completed square form, the coordinates of the vertex are easily identifiable. For example, for the function $y = 2(x + 3)^2 - 2$, the coordinates of the vertex are $(-3, -2)$.

If the equation for a quadratic function is in standard form

($y = ax^2 + bx + c$), the coordinates of the vertex are not easily identifiable. In order to identify the coordinates of the vertex, you must convert the standard form to completed square form. This section shows you an alternate method if you have trouble completing the square. If the equation for a quadratic function is written in standard form, the coordinates of the vertex may be found by using a formula.



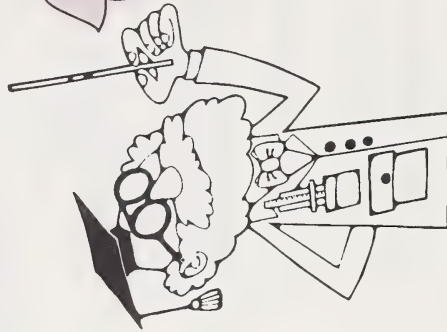
For the quadratic function $y = ax^2 + bx + c$, the coordinates of the vertex may be found by substituting the values a , b , and c into a formula.



For $y = ax^2 + bx + c$, the coordinates of the vertex may be found by using the formula

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a} \right),$$

where a is the coefficient of x^2 ,
 b is the coefficient of x ,
and c is the constant term.



Follow the next examples to see how to use this formula.

Example 22

Find the coordinates of the vertex for each quadratic function that follows:

- $y = 2x^2 + 12x + 16$

Solution:

$$y = 2x^2 + 12x + 16$$

Identify the values of a , b , and c .

$$a = 2, b = 12, c = 16$$

$$\begin{aligned}\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right) &= \left(-\frac{12}{2 \times 2}, \frac{4 \times 2 \times 16 - 12^2}{4 \times 2}\right) \\ &= \left(-\frac{12}{4}, \frac{128 - 144}{8}\right) \\ &= \left(-3, \frac{-16}{8}\right) \\ &= (-3, -2)\end{aligned}$$

The vertex is at $(-3, -2)$.

- $y = -3x^2 + 2x - 4$

Solution:

$$y = -3x^2 + 2x - 4$$

Identify the values of a , b , and c .

$$a = -3, b = 2, c = -4$$

$$\begin{aligned}\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right) &= \left(\frac{-2}{2(-3)}, \frac{4(-3)(-4) - 2^2}{4(-3)}\right) \\ &= \left(\frac{-2}{-6}, \frac{48 - 4}{-12}\right) \\ &= \left(\frac{1}{3}, \frac{44}{-12}\right) \\ &= \left(\frac{1}{3}, -\frac{11}{3}\right)\end{aligned}$$

The vertex is at $\left(\frac{1}{3}, -\frac{11}{3}\right)$.

Example 23

Find the coordinates of the vertex for the following:

$$\bullet y = 3x^2 + 12x$$

Solution:

$$y = 3x^2 + 12x$$

$$a = 3, b = 12, c = 0$$

$$\begin{aligned}\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right) &= \left(-\frac{12}{2 \times 3}, \frac{4 \times 3 \times 0 - 12^2}{4 \times 3}\right) \\ &= \left(-\frac{12}{6}, \frac{0 - 144}{12}\right) \\ &= (-2, -12)\end{aligned}$$

The vertex is at $(-2, -12)$.

$$\bullet y = -2x^2 - 5$$

Solution:

$$y = -2x^2 - 5$$

$$a = -2, b = 0, c = -5$$

$$\begin{aligned}\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right) &= \left(\frac{0}{2(-2)}, \frac{4(-2)(-5) - 0^2}{4(-2)}\right) \\ &= \left(\frac{0}{-4}, \frac{40 - 0}{-8}\right) \\ &= (0, -5)\end{aligned}$$

The vertex is at $(0, -5)$.

Try the following problems.

For each quadratic function that follows, identify the values of a , b , and c . Find the coordinates of the vertex.

1. $y = -3x^2 + 2x - 1$
2. $y = 5x^2 - 15x$
3. $y = 4x^2 + x - 3$
4. $y = 2x^2 + 6$
5. $y - 3 + 2x^2 = -6x$
6. $y + 3 = 4x + 3x^2$



For solutions to Extra Help, turn to Appendix A, Topic 3.



Extensions

If you found characteristics of quadratic functions interesting and would like to improve your skills, try the following question.

Put the quadratic function $y = ax^2 + bx + c$ into completed square form. After it is in completed square form, write a formula to find the maximum or minimum of the function in terms of a , b , and c . Also, write the equation for the axis of symmetry.



For solutions to Extensions, turn to **Appendix A, Topic 3**.

Note: If you are interested in how this formula works, see the **Extra Help** section in **Topic 3**.

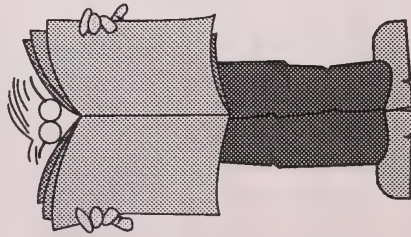


Topic 4 Solving Quadratic Equations



Introduction

Finding the y -intercept for a quadratic function is a simple matter. You simply replace every x with a zero and solve for y . The resulting y -value is the y -intercept. In this topic you will be shown how to find the x -intercepts for a quadratic function using a similar method.



What Lies Ahead

Throughout the topic you will learn to

1. find the x -intercepts for a quadratic function by solving the corresponding quadratic equation

Now that you know what to expect, turn the page to begin your study of solving quadratic equations.



Exploring Topic 4

Activity 1



Find the x -intercepts for a quadratic function by solving the corresponding quadratic equation.



Since the y -intercept for a function is the point where the graph crosses the y -axis (vertical axis), the x -intercept must be the point where the graph crosses the x -axis (horizontal axis).

- What is the x -intercept of a function?

In the last topic you learned that the y -intercept for a function is the point where the graph of the function crosses the y -axis. Since the value of x at this point is zero, you can find the y -intercept by substituting zero for x in the equation and solving for y .

For example, the y -intercept for the equation $y = 3x^2 - 5x + 6$ can be found by substituting zero wherever you see an x .

$$y = 3(0)^2 - 5(0) + 6$$

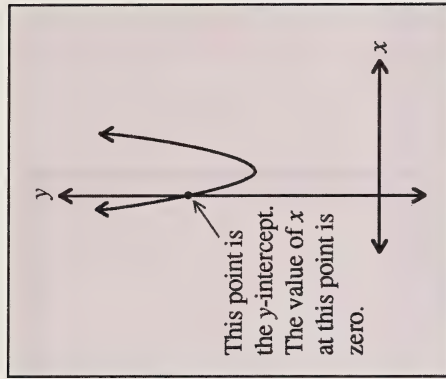
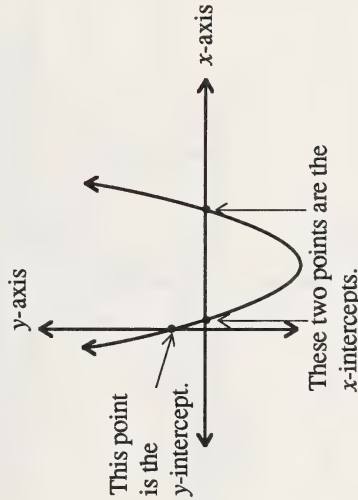
$$y = 3(0) - 0 + 6$$

$$y = 0 - 0 + 6$$

$$y = 6$$

The y -intercept is 6. This means that the graph of $y = 3x^2 - 5x + 6$ crosses the y -axis at the point $(0, 6)$.

Use your understanding of the previous material to answer the following questions about the x -intercept of a quadratic function.

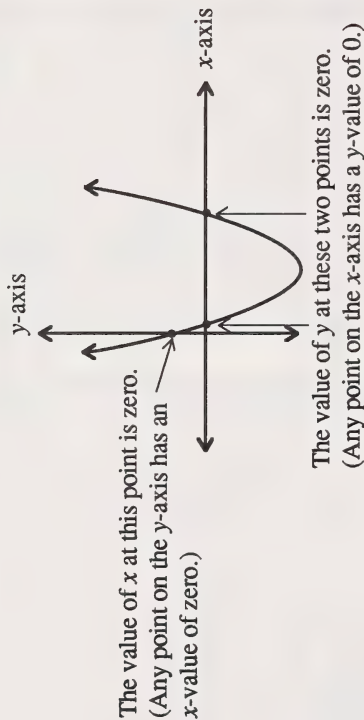


Some functions have more than one x -intercept.

- What is the value of y at the x -intercept?



Since the x -intercept is the point at which the graph crosses the x -axis, the value of y at the x -intercept must be zero. The x -intercepts are also called the **zeros** of the function.



- If you know the equation of a function, how can you determine the x -intercept?



To find the y -intercept, you let x equal zero and then solve for y . To find the x -intercept, you let y equal zero and solve for x .

Example 1

Find the x - and y -intercepts for the function

$$y = x^2 - 8x + 15.$$

Solution:

To find the y -intercept, let $x = 0$.

$$y = (0)^2 - 8(0) + 15$$

$$y = 0 - 0 + 15$$

$$y = 15$$

The y -intercept is 15. The graph crosses the y -axis at $(0, 15)$.

For the y -intercept, the x -coordinate is 0.

To find the x -intercept, let $y = 0$.

$$0 = x^2 - 8x + 15$$

$$0 = (x - 3)(x - 5)$$

$$x - 3 = 0 \text{ or } x - 5 = 0$$

$$x = 3 \qquad x = 5$$

The x -intercepts are 3 and 5. The graph crosses the x -axis at $(3, 0)$ and $(5, 0)$.

For the x -intercepts, the y -coordinates are 0.



For the quadratic function

$y = ax^2 + bx + c$, where $a \neq 0$ and a , b , and c are real numbers, the x -intercepts can be found by letting y equal zero and solving for x in the equation $0 = ax^2 + bx + c$.

The equation $ax^2 + bx + c = 0$ has a special name. Any equation of the form

$ax^2 + bx + c = 0$, $a \neq 0$ is called a **quadratic equation** in one variable.

All quadratic equations in one variable are of degree two.

They contain one variable with an exponent of two.



Another method that you could use to find the x - and y -intercepts would be to sketch the graph of the function and identify the x - and y -intercepts from the graph.

Note the difference between a quadratic function and its corresponding quadratic equation.

$$y = ax^2 + bx + c \text{ (quadratic function)}$$

$$0 = ax^2 + bx + c \text{ (corresponding quadratic equation)}$$

To find the x -intercepts for a quadratic function, solve the corresponding quadratic equation. To solve a quadratic equation, you must find a numerical value for the variable that makes the equation true. This value is called a **root** of the equation. For example, 3 is a root of the equation $x^2 + 4x - 21 = 0$ because if you replace x with 3, the equation is true ($LS = RS$).

Check:

$$x = 3$$

LS	RS
$x^2 + 4x - 21$	0
$(3)^2 + 4(3) - 21$	0
$9 + 12 - 21$	0
0	0
LS = RS	

The x -intercepts, zeros, and roots all refer to the same values for a given quadratic function or its corresponding quadratic equation. When you have a quadratic function, the values where the graph crosses the x -axis are called the x -intercepts or zeros. When you have the corresponding quadratic equation, the values where the graph crosses the x -axis are called the roots of the equation.

Before you solve for the roots of a quadratic equation, you should write the equation in standard form which is $ax^2 + bx + c = 0$ (or $0 = ax^2 + bx + c$). In standard form, all the terms are on one side and the zero is on the other side.

Look at the next example.

Example 2

Put each quadratic equation that follows in standard form and identify the values of a , b , and c .

- $3x^2 = 2x - 1$

Solution:

$$3x^2 - 2x + 1 = 0$$

$$a = 3, b = -2, c = 1$$

- $1 = -2x^2$

Solution:

$$2x^2 + 1 = 0$$

$$a = 2, b = 0, c = 1$$

- $5x = -3x^2$

Solution:

$$3x^2 + 5x = 0$$

$$a = 3, b = 5, c = 0$$

The x -intercepts for a quadratic function are found by calculating the roots of the corresponding quadratic equation. The following pages show you how to solve quadratic equations. You should relate the type of equation to the method used to solve it. A good understanding of factoring is essential.

Type 1

Quadratic equations can be of the form $ax^2 + c = 0$ ($b = 0$). There is no x -term in this type. There are two methods that you may use to solve this type.

The Method 1 solution to this type relies on your ability to factor the difference of two squares.

$$(x^2 - 25) = (x + 5)(x - 5)$$

$$(x^2 - 9) = (x + 3)(x - 3)$$

You must also make use of the fact that if $(a)(b) = 0$, then $a = 0$ or $b = 0$. For example if $(x + 7)(x - 5) = 0$, then the following is true:

$$x + 7 = 0 \quad \text{or} \quad x - 5 = 0$$

$$x = -7 \quad x = 5$$

Look at the next example.

Example 3

Solve each quadratic equation that follows.

- $x^2 - 9 = 0$

Solution:

$$x^2 - 9 = 0$$

Method 1

$$(x + 3)(x - 3) = 0 \quad (\text{Factor.})$$

$$x + 3 = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = -3 \quad x = 3$$

The roots are -3 and 3 .

Method 2

$$x^2 = 9$$

$$x = \pm\sqrt{9}$$

$$x = \pm 3$$

$$\bullet 3x^2 - 12 = 0$$

Solution:

$$3x^2 - 12 = 0$$

Method 1

$$3(x^2 - 4) = 0 \quad (\text{Factor.})$$

$$3(x+2)(x-2) = 0$$

$$x+2=0 \text{ or } x-2=0$$

$$x = -2 \quad x = 2$$

The roots are -2 and 2 .

$$\bullet x^2 - 8 = 0$$

Solution:

$$x^2 - 8 = 0$$

Method 2

$$3x^2 = 12$$

$$x^2 = \frac{12}{3}$$

$$x^2 = 4$$

$$x = \pm\sqrt{4}$$

$$x = \pm 2$$

Method 1

$$(x + \sqrt{8})(x - \sqrt{8}) = 0 \quad (\text{Factor.})$$

$$x + \sqrt{8} = 0 \text{ or } x - \sqrt{8} = 0$$

$$x = -\sqrt{8} \quad x = \sqrt{8}$$

$$x = -2\sqrt{2} \quad x = 2\sqrt{2}$$

$$(\text{Recall that } \sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}.)$$

The roots are $-2\sqrt{2}$ and $2\sqrt{2}$.

$$\bullet 4x^2 - 25 = 0$$

Solution:

$$4x^2 - 25 = 0$$

Method 1

$$(2x+5)(2x-5) = 0 \quad (\text{Factor.})$$

$$2x+5=0 \text{ or } 2x-5=0$$

$$2x = -5 \quad 2x = 5$$

$$x = -\frac{5}{2} \quad x = \frac{5}{2}$$

The roots are $-\frac{5}{2}$ and $\frac{5}{2}$.

Method 2

$$x^2 = 8$$

$$x = \pm\sqrt{8}$$

$$x = \pm 2\sqrt{2}$$

Method 2

$$4x^2 = 25$$

$$x^2 = \frac{25}{4}$$

$$x = \pm\sqrt{\frac{25}{4}}$$

$$x = \pm\frac{5}{2}$$

Quadratic equations like $x^2 + 9 = 0$ do not have a real-number root.

$$x^2 + 9 = 0$$

$$x^2 = -9$$

$$x = \pm\sqrt{-9} \quad (\text{This is defined as a complex number.})$$

Type 2

Quadratic equations can be of the form

$ax^2 + bx = 0$ ($c = 0$). There is no constant term in this type. This type is solved by factoring.

Look at the following example.

Example 4

Solve each quadratic equation that follows.

$$\bullet \quad x^2 - 3x = 0$$

Solution:

$$x^2 - 3x = 0$$

$$x(x - 3) = 0 \quad (\text{Factor.})$$

$$x = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = 3$$

The roots are 0 and 3.

$$\bullet \quad -2x^2 + 5x = 0$$

Solution:

$$-2x^2 + 5x = 0$$

$$-x(2x - 5) = 0 \quad (\text{Factor.})$$

$$-x = 0 \quad \text{or} \quad 2x - 5 = 0$$

$$x = 0$$

$$2x = 5$$

$$x = \frac{5}{2}$$

The roots are 0 and $\frac{5}{2}$.

$$\bullet \quad 3x^2 - 15x = 0$$

Solution:

$$3x^2 - 15x = 0$$

$$3x(x - 5) = 0 \quad (\text{Factor.})$$

$$3x = 0 \quad \text{or} \quad x - 5 = 0$$

$$x = \frac{0}{3} \quad x = 5$$

$$x = 0$$

The roots are 0 and 5.

The square root of a negative number such as $\pm\sqrt{-9}$ is not a real number.

Type 3

Quadratic equations can be of the form

$ax^2 + bx + c = 0$, where a , b , and c are nonzero numbers. This type is solved by factoring.

Look at the following example.

Example 5

Solve each quadratic equation that follows.

- $x^2 - 8x + 15 = 0$

Solution:

$$x^2 - 8x + 15 = 0$$

$$(x - 3)(x - 5) = 0 \quad (\text{Factor.})$$

$$x - 3 = 0 \quad \text{or} \quad x - 5 = 0$$

$$x = 3 \qquad x = 5$$

The roots are 3 and 5.

- $3x^2 - 9x + 6 = 0$

Solution:

$$3x^2 - 9x + 6 = 0$$

$$3(x^2 - 3x + 2) = 0 \quad (\text{Factor.})$$

$$3(x - 2)(x - 1) = 0$$

$$x - 2 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = 2 \qquad x = 1$$

The roots are 2 and 1.

- $6x^2 - 7x - 3 = 0$

Solution:

$$6x^2 - 7x - 3 = 0$$

$$(3x + 1)(2x - 3) = 0 \quad (\text{Factor.})$$

$$3x + 1 = 0 \quad \text{or} \quad 2x - 3 = 0$$

$$3x = -1 \qquad 2x = 3$$

$$x = -\frac{1}{3} \qquad x = \frac{3}{2}$$

The roots are $-\frac{1}{3}$ and $\frac{3}{2}$.

If you have trouble when factoring trinomials, see the **Extra Help** section.

To find the x -intercepts for a quadratic function, you find the roots of the corresponding quadratic equation. The following examples show you how this is done.

Example 6

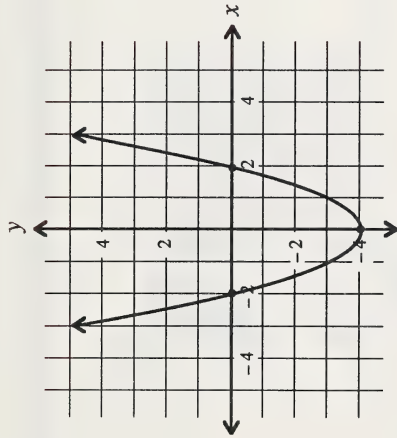
Draw the graph of each function that follows by using a table of values or a computer. Identify the x -intercepts from the graph, and then determine the x -intercepts for each function algebraically by letting y equal zero and solving for the roots of the corresponding quadratic equation. Compare the two results.

- $y = x^2 - 4$

Solution:

$$y = x^2 - 4$$

x	-3	-1	0	1	3
y	5	-3	-4	-3	5



The graph crosses the x -axis at $(-2, 0)$ and $(2, 0)$.

$$y = x^2 - 4 \quad (\text{Let } y = 0.)$$

$$0 = x^2 - 4$$

$$0 = (x + 2)(x - 2)$$

$$x + 2 = 0 \text{ or } x - 2 = 0$$

$$x = -2 \quad x = 2$$

The roots are -2 and 2 .

(These roots are real and unequal.)

If you examine the graph you see that the parabola has two x -intercepts. The x -intercepts are at -2 and 2 . The equation $0 = x^2 - 4$ has two roots, -2 and 2 .

To find the x -intercepts, let $y = 0$.

The corresponding quadratic equation for the function $y = x^2 - 4$ is $x^2 - 4 = 0$.

Some quadratic equations have roots that are real and equal.

• $y = x^2 + 4x + 3$

Solution:

$y = x^2 + 4x + 3$

x	-5	-4	-2	0	1
y	8	3	-1	3	8

$y = x^2 + 4x + 3$ (Let $y = 0$.)

$0 = x^2 + 4x + 3$

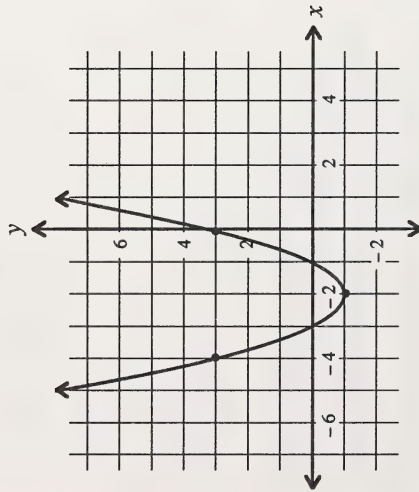
$0 = (x+3)(x+1)$

$x+3=0$ or $x+1=0$

$x = -3$ $x = -1$

The roots are -3 and -1 .
(The roots are real and unequal.)

From the graph you can see that the x -intercepts are located at -1 and -3 . The roots of the corresponding equation are also -1 and -3 .



The x -intercepts are -3 and -1 .

Some quadratic functions only have one x -intercept. How many roots will the corresponding quadratic equation have?



The corresponding quadratic equation for the quadratic function $y = x^2 + 4x + 3$ is $0 = x^2 + 4x + 3$.

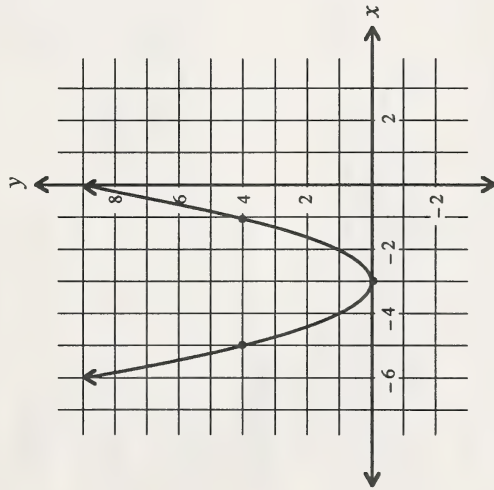
Example 7

Draw the graph of $y = x^2 + 6x + 9$ by using a table of values or a computer. Then algebraically determine the x -intercepts for the graph of the function. Compare the two results.

Solution:

$$y = x^2 + 6x + 9$$

x	-6	-5	-3	-1	0
y	9	4	0	4	9



The x -intercept is -3 .

$$y = x^2 + 6x + 9 \quad (\text{Let } y = 0.)$$

$$0 = x^2 + 6x + 9$$

$$0 = (x + 3)(x + 3)$$

$$x + 3 = 0 \text{ or } x + 3 = 0$$

$$x = -3 \quad x = -3$$

The root is -3 .

(There are two real roots, but they are equal.)

Again the x -intercept of the graph has the same value as the root of the equation.

From the previous examples you can see that if a quadratic function has one x -intercept, the corresponding quadratic equation will have two real and equal roots. Usually, if the roots are equal, you say that there is only one root.

If a quadratic function has one x -intercept, the parabola will just touch the x -axis at this point. The vertex of the parabola is located at this point.

Some quadratic functions do not have any x -intercepts. What kind of roots will the corresponding quadratic equation have? Look at the next example to find out.

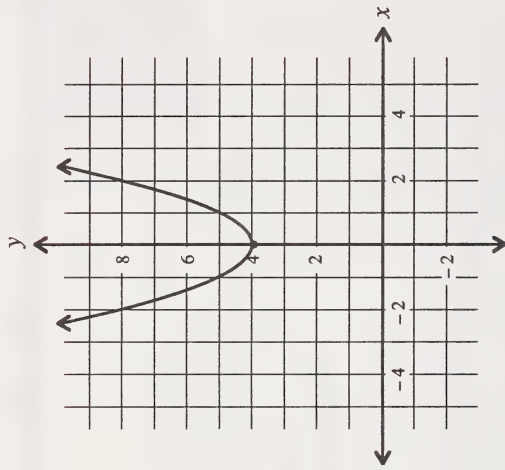
Example 8

Draw the graph of $y = x^2 + 4$ and algebraically determine the x -intercepts for the graph. Write a conclusion regarding the roots and the x -intercepts.

Solution:

$$y = x^2 + 4$$

x	-2	-1	0	1	2
y	8	5	4	5	8



There are no x -intercepts.

$$y = x^2 + 4 \quad (\text{Let } y = 0.)$$

$$0 = x^2 + 4 \quad \text{or} \quad x^2 + 4 = 0$$

$$x^2 = -4$$

$$x = \pm \sqrt{-4}$$

Note that $\sqrt{-4}$ is not a real number.

There are no real roots.

From the previous examples you can conclude the following:

- If a quadratic function does not have any x -intercepts, the corresponding equation does not have any real roots.
- If a quadratic equation does not have any real roots, the corresponding function does not have any x -intercepts.

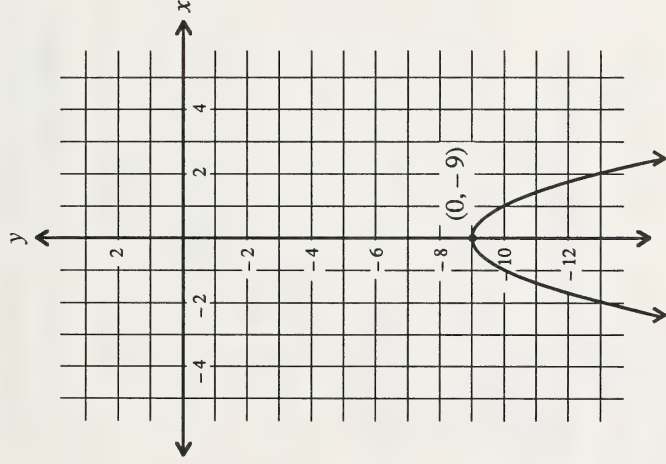
You can predict that the graph of $y = x^2 + 4$ does not have any x -intercepts by using translations.

The graph of $y = x^2 + 4$ is congruent to the graph of $y = x^2$ translated four units upward. Its vertex is at $(0, 4)$ and since the parabola opens upward, it cannot have any x -intercepts.

Using the same reasoning, you can predict that the graph of $y = -x^2 - 9$ does not have any x -intercepts.

The graph of $y = -x^2 - 9$ is congruent to the graph of $y = -x^2$ translated nine units downward. The vertex is at $(0, -9)$ and since the parabola opens downward, it cannot have any x -intercepts.

The graph and a confirmation of this prediction are shown as follows.



$$y = -x^2 - 9 \quad (\text{Let } y = 0.)$$

$$-x^2 - 9 = 0$$

$$-x^2 = 9$$

$$x^2 = -9$$

$$x = \pm\sqrt{-9}$$

If a parabola does not have any x -intercepts, it must lie entirely above or below the x -axis.

$$y = -x^2 - 9$$

The negative sign in front of the x^2 means the parabola opens downward.

Note that $\sqrt{-9}$ is not a real number. This equation does not have any real roots; therefore, the corresponding function does not have any x -intercepts.

From the previous examples you can conclude the following:

- The x -intercepts for a quadratic function have the same value as the roots of the corresponding quadratic equation.
- The number of x -intercepts and the number of roots are the same.



The following are two methods that you may use to find the x -intercepts for a quadratic function:

- Graph the function and identify the x -intercepts from the graph.
- Solve for the roots of the corresponding quadratic equation.

Many graphs have x -intercepts that are not easily identifiable. For example, x -intercepts like $-\frac{2}{5}$ or $\frac{1}{3}$ would be rather difficult to identify from a graph. For this reason, it is advisable to find the x -intercepts for a quadratic function by finding the roots of the corresponding quadratic equation. This method allows you to calculate the exact value of nonintegral x -intercepts. Study the following example.

Example 9

Find the x -intercept for each function that follows.

- $y = 15x^2 + x - 2$

Solution:

$$y = 15x^2 + x - 2 \quad (\text{Let } y = 0.)$$

$$0 = 15x^2 + x - 2$$

$$0 = (3x - 1)(5x + 2)$$

$$3x - 1 = 0 \quad \text{or} \quad 5x + 2 = 0$$

$$3x = 1 \qquad 5x = -2$$

$$x = \frac{1}{3} \qquad x = -\frac{2}{5}$$

The roots are $\frac{1}{3}$ and $-\frac{2}{5}$. The x -intercepts for the graph occur at $\frac{1}{3}$ and $-\frac{2}{5}$.

Nonintegral x -intercepts are x -intercepts that are not integers.

$$\bullet y = 12x^2 + 13x - 14$$

Solution:

$$y = 12x^2 + 13x - 14 \quad (\text{Let } y = 0.)$$

$$0 = 12x^2 + 13x - 14$$

$$0 = (3x - 2)(4x + 7)$$

$$3x - 2 = 0 \text{ or } 4x + 7 = 0$$

$$3x = 2 \quad 4x = -7$$

$$x = \frac{2}{3} \quad x = -\frac{7}{4}$$

The roots are $\frac{2}{3}$ and $-\frac{7}{4}$. The x-intercepts for the graph occur at $\frac{2}{3}$ and $-\frac{7}{4}$.

As you can see, in order to find the roots of quadratic equations, you must be competent at factoring. The exercises that follow contain many factoring problems. Be sure that you can factor each type.

Try the following questions. Graph paper is provided in **Appendix B**.

1. To find the y-intercept for a quadratic function, substitute a zero for x in the function and solve for y. What is the logic behind this process?
2. To find the x-intercepts for a quadratic function, substitute a zero for y in the function and solve for x. What is the logic behind this process?

$$3. \text{ Find the x- and y-intercepts for the function } y = x^2 - 2x - 8.$$

$$4. \text{ Show the difference between a quadratic function and a quadratic equation.}$$

$$5. \text{ Determine whether or not } -7 \text{ is a root of the equation } x^2 + 4x - 21 = 0.$$

$$6. \text{ Write each quadratic equation that follows in standard form } (ax^2 + bx + c = 0).$$

$$a. \quad 1 + 2x^2 = 3x$$

$$b. \quad 3x = -7x^2$$

$$c. \quad 7 + 2x^2 = 0$$

$$7. \text{ Find the roots for each quadratic equation that follows. Do any five.}$$

$$a. \quad x^2 - 25 = 0$$

$$b. \quad x^2 = 1$$

$$c. \quad x^2 - 3 = 0$$

$$d. \quad x^2 + 4 = 0$$

$$e. \quad x^2 = -25$$

$$f. \quad 4x^2 - 1 = 0$$

$$g. \quad 3x^2 - 27 = 0$$

$$h. \quad 3x^2 = 0$$

$$i. \quad x^2 - 2 = 0$$

$$j. \quad -2x^2 - 8 = 0$$

8. Find the roots for each quadratic equation that follows. Do any five.

a. $3x^2 - 12x = 0$

b. $x^2 + 7x = 0$

c. $-x^2 + 5x = 0$

d. $x^2 = 4x$

e. $+5x = -x^2$

f. $2x^2 = 16x$

g. $-3x^2 - 7x = 0$

h. $2x^2 + 5x = 0$

9. Solve each quadratic equation that follows by factoring. Do any five.

a. $x^2 - 3x + 2 = 0$

b. $x^2 = 2x - 1$

c. $0 = x^2 - 7x + 12$

d. $x^2 - x - 56 = 0$

e. $x^2 - 3x = 28$

f. $0 = x^2 - 5x - 14$

g. $x^2 + 3x - 4 = 0$

h. $-x^2 - x = -20$

i. $x^2 + 3x + 5 = 0$

10. Solve each quadratic equation that follows. Do any five.

a. $4x^2 - 4x + 1 = 0$

b. $2x^2 + 7x + 6 = 0$

c. $2x^2 + 11x = 6$

d. $6x^2 + 23x + 20 = 0$

e. $4x^2 - 7x + 3 = 0$

f. $5x^2 - 7x - 6 = 0$

g. $7x^2 - 2x - 5 = 0$

h. $5x^2 - 2x - 7 = 0$

i. $-5x^2 = -3 + 14x$

11. Roughly sketch the graph of a quadratic function that only has one x -intercept.

If you have difficulty factoring these, see the **Extra Help** section.

12. If a quadratic equation does not have any real roots, what do you know about the graph of the corresponding quadratic function?
13. By using translations, show that the graph of $y = x^2 + 3$ does not have any x -intercepts. Sketch this graph.
14. Find the x -intercept(s) for the following quadratic functions by finding the roots of the corresponding quadratic equations. Do any five.

a. $y = 3x^2 - 3$

b. $y = 3x^2 - 3x$

c. $y = x^2 - 3x - 4$

d. $y = 2x^2 - 3x - 5$

e. $y = x^2 + 9$

f. $y = -x^2 - 12x$

g. $y = x^2 + x - 30$

h. $y = x^2 - 8x + 16$



For solutions to **Activity 1**, turn to **Appendix A**,
Topic 4.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.



Extra Help

Do either Part A or Part B or both.



Part A

Use the computer program *Solving Quadratic Equations*¹ for extra help with factoring trinomials and solving for the roots of quadratic equations.



Part B

If you have trouble with factoring trinomials, then this section may benefit you. Follow the development and do the exercises at the end.

¹ *Solving Quadratic Equations* is a title of Microcomputer Workshops.

Trinomials may be divided into two different types. Each type can be factored using a different method.

Type 1

Trinomials can be of the form $ax^2 + bx + c$, where $a = 1$. Examples of this type are $x^2 + 5x + 6$ and $x^2 - 3x + 2$. The coefficient in front of the x^2 is one.

Type 2

Trinomials can be of the form $ax^2 + bx + c$, where $a \neq 1$. Examples of this type are $2x^2 - 5x + 3$ and $5x^2 + 2x - 3$. The coefficient in front of the x^2 is not one.

Factoring Type 1 Trinomials

To factor this type, use the inspection method. Follow the next example.

Example 10

Factor each trinomial that follows.

$$\bullet x^2 + 5x + 6$$

Solution:

$$x^2 + 5x + 6$$

If trinomials are factorable, they can be factored into two binomials.

$$(\quad)(\quad)$$

Write down two empty sets of parentheses representing the binomials.

$$(x \quad)(x \quad)$$

The first term in each binomial can be found by taking the square root of the first term of the trinomial.

$$\sqrt{x^2} = x$$

$$(x+2)(x+3)$$

The second term in each binomial is determined by finding two numbers whose product is the last term of the trinomial and whose sum is the coefficient of the middle term of the trinomial.

$$(2)(3) = 6 \text{ and } 2 + 3 = 5$$

The two numbers are 2 and 3. Write 3 in one binomial and 2 in the other.

$$(x+2)(x+3)$$

The trinomial is factored.

$$\bullet x^2 - 3x + 2$$

Solution:

$$x^2 - 3x + 2$$

Make sure the trinomial is in standard form.

$$(\quad)(\quad)$$

Write two empty sets of parentheses.

$$(x \quad)(x \quad)$$

Since $\sqrt{x^2} = x$, put an x as the first term in the parentheses.

$$(x-2)(x-1)$$

Find two numbers that multiply to give a product of 2 and add to give a sum of -3 .

$$(-2)(-1) = 2 \text{ and } (-2) + (-1) = -3$$

The two numbers are -2 and -1 ; therefore, put -2 in one binomial and -1 in the other.

$$(x-2)(x-1)$$

The trinomial is factored.

$$\bullet x^2 - 5x - 14$$

Solution:

$$x^2 - 5x - 14$$

$$(\quad)(\quad)$$

Write down two empty sets of parentheses.

$$(x \quad)(x \quad)$$

Since $\sqrt{x^2} = x$, put an x as the first term in the parentheses.

$$(x - 7)(x + 2)$$

Find two numbers whose product is -14 and whose sum is -5 .

$$(-7)(2) = -14 \text{ and } (-7) + (2) = -5$$

The two numbers are -7 and 2 .

Write -7 in one set of parentheses and 2 in the other set of parentheses.

$$(x - 7)(x + 2)$$

The trinomial is factored.

Factoring Type 2 Trinomials

To factor this type, use the decomposition method of factoring. Follow the next example.

Example 11

Factor the following trinomials.

$$\bullet 2x^2 - 7x + 6$$

Solution:

$$2x^2 - 7x + 6$$

The trinomial must be in standard form.

Multiply the coefficient of the x^2 term by the constant term.

Find two numbers that multiply to give you 12 and add to give you the coefficient of the x -term which is -7 .

The two numbers are -3 and -4 .

Rewrite the trinomial by

decomposing $-7x$ into $-3x + (-4x)$.

Place parentheses around the first two terms and the last two terms.

Remove the G.C.F. from each set of parentheses.

$$\begin{array}{c} \swarrow \searrow \\ 2x^2 - 7x + 6 \\ \downarrow \quad \downarrow \\ (2)(6) = 12 \end{array}$$

$$(-3)(-4) = 12$$

$$(-3) + (-4) = -7$$

$$2x^2 - 7x + 6$$

$$2x^2 - 3x + (-4x) + 6$$

$$(2x^2 - 3x) + (-4x + 6)$$

$$x(2x - 3) + [-2(2x - 3)]$$

$$(2x-3)[x+(-2)]$$

or $(2x-3)(x-2)$

$$\bullet 5x^2 - 2x - 3$$

Solution:

$$5x^2 - 2x - 3$$

$$\swarrow \searrow$$

$$(5)(-3) = -15$$

Multiply the coefficient of the first term and the constant term.

$$(-5)(3) = -15$$

$$-5 + 3 = -2$$

Find two numbers whose product is -15 and whose sum is -2 . (The coefficient of the x -term is -2 .) The two numbers are -5 and 3 .

$$5x^2 - 2x - 3$$

$$5x^2 - 5x + 3x - 3$$

$$(5x^2 - 5x) + (3x - 3)$$

$$5x(x-1) + 3(x-1)$$

$$(x-1)(5x+3)$$

Factor out the common binomial and the trinomial is factored.

$$\bullet 3x^2 + 2x - 8$$

Solution:

$$3x^2 + 2x - 8$$

$$\swarrow \searrow$$

$$(3)(-8) = -24$$

$$(6)(-4) = -24$$

$$6 + (-4) = 2$$

The two numbers whose product is -24 and whose sum is 2 are 6 and -4 .

$$3x^2 + 2x - 8$$

$$3x^2 + 6x + (-4x) - 8$$

$$(3x^2 + 6x) + (-4x - 8)$$

Decompose $2x$ into $6x + (-4x)$.

Place parentheses around each pair of terms.

$$3x(x+2) - 4(x+2)$$

$$(x+2)(3x-4)$$

Remove the G.C.F.

Factor out the common binomial.

Use the previous examples as a guide when doing the following exercises.

1. Factor each Type 1 trinomial that follows.

- a. $x^2 - 5x - 24$
- b. $x^2 - x - 30 = 0$
- c. $x^2 - 2x + 1$
- d. $x^2 - 3x - 40$
- e. $x^2 + x - 20$
- f. $x^2 + 9x + 14$
- g. $x^2 - 5x - 6$
- h. $x^2 + 2x - 8$
- i. $x^2 + 3x - 8$

2. Factor each Type 2 trinomial that follows.

- a. $6x^2 + 13x + 6$
- b. $6x^2 - 13x + 6$
- c. $6x^2 - 5x - 6$
- d. $6x^2 + 5x - 6$
- e. $6x^2 + 11x + 5$
- f. $6x^2 - 17x + 5$
- g. $6x^2 + x - 5$
- h. $6x^2 - 29x - 5$
- i. $6x^2 - 2x - 5$

For solutions to **Extra Help**, turn to **Appendix A**, **Topic 4**.



Extensions

A quadratic function is of degree two. The graph of a quadratic function may have two x -intercepts at most.

A cubic function is of degree three. The graph of a cubic function may have three x -intercepts at most.

Do the following exercise. Graph paper is provided in **Appendix B**.

Draw the graph of the cubic function $y = x^3 + x^2 - 6x$ by using a table of values or a computer. Identify the x -intercepts from the graph and then algebraically find the roots of the corresponding cubic equation. Do the roots and the x -intercepts have the same value?



For solutions to **Extensions**, turn to **Appendix A**, **Topic 4**.

Topic 5 The Quadratic Formula



Introduction

In the last topic you learned that quadratic equations can be solved by factoring. However, many quadratic equations are not factorable. How do you find the roots for these equations? In this topic you will be shown a formula that you may use to find the roots of any quadratic equation.



What Lies Ahead

Throughout the topic you will learn to

1. solve quadratic equations by using the quadratic formula

Now that you know what to expect, turn the page to begin your study of the quadratic formula.



Exploring Topic 5

Activity 1



Solve quadratic equations by using the quadratic formula.

In the last topic you learned two ways to find the x -intercepts for a quadratic function.

The x -intercepts can be found by

- drawing the graph from a table of values or on a computer, and identifying the x -intercepts from the graph
- calculating the roots of the corresponding quadratic equation (In most cases, this involves factoring)

The next example illustrates an interesting situation. Follow it carefully.

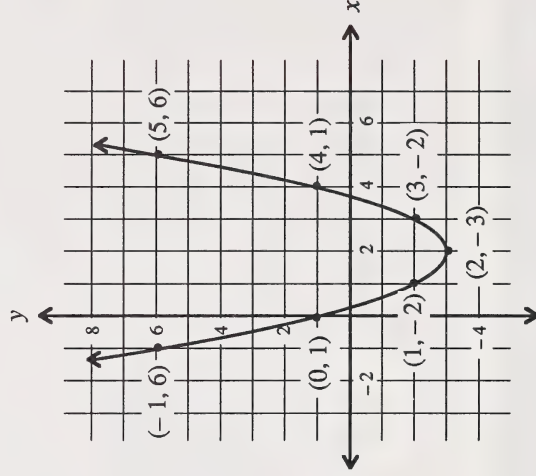
Example 1

Draw the graph of $y = x^2 - 4x + 1$. Identify the x -intercepts from the graph. Can the x -intercepts be determined algebraically?

Solution:

$$y = x^2 - 4x + 1$$

x	-1	0	1	2	3	4	5
y	6	1	-2	-3	-2	1	6



From the graph, there are two x -intercepts whose values are approximately 0.25 and 3.75. Since the x -intercepts are approximately 0.25 and 3.75, the roots of the equation also must be approximately 0.25 and 3.75.

If you try to determine the x -intercepts algebraically, you will soon realize that the following equation is not readily factorable.

$$0 = x^2 - 4x + 1$$

$$0 = (x?)(x?)$$

You cannot find two integers that multiply to give 1 and add to give a sum of -4 .

The approximate value for the roots of the equation can be obtained from the graph of the function. It is difficult to obtain an exact value from the graph. Since the equation is not factorable, there must be another way to determine the exact roots of the equation. If a quadratic equation is not factorable, the roots may be calculated by **completing the square**. The following example shows you how to find the roots of a quadratic equation by completing the square. You learned how to complete the square for a quadratic function in **Topic 3**; therefore, this should not be too difficult.

Example 2

Find the roots of the equation $x^2 - 4x + 1 = 0$ by completing the square.

Solution:

$$x^2 - 4x + 1 = 0$$

Put parentheses around the terms containing x^2 and x .

$$(x^2 - 4x) + 1 = 0$$

Complete the square inside the parentheses. Also subtract 4 outside the parentheses.

$$\begin{aligned} \left[\frac{1}{2} \times (-4) \right]^2 &= (-2)^2 \\ &= 4 \\ (x^2 - 4x + 4) + 1 - 4 &= 0 \end{aligned}$$

Change the trinomial inside the parentheses to a perfect square. Now the equation is in completed square form.

$$(x-2)^2 - 3 = 0$$

Solve for x .

Isolate the binomial and take the square root of both sides.

$$(x-2)^2 = 3$$

$$x-2 = \pm\sqrt{3}$$

$$x = 2 \pm \sqrt{3} \quad (\text{Isolate the } x.)$$

Note that $x = 2 \pm \sqrt{3}$ implies that $x = 2 + \sqrt{3}$ or $x = 2 - \sqrt{3}$.

The two roots are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

The validity of the two roots may be verified by substituting back into the original equation.

Check:

$$x = 2 + \sqrt{3}$$

LS	RS
$x^2 - 4x + 1$	0
$(2 + \sqrt{3})^2 - 4(2 + \sqrt{3}) + 1$	0
$4 + 4\sqrt{3} + 3 - 8 - 4\sqrt{3} + 1$	0
$0 + 0$	0
0	0
LS =	RS

$$x = 2 - \sqrt{3}$$

LS	RS
$x^2 - 4x + 1$	0
$(2 - \sqrt{3})^2 - 4(2 - \sqrt{3}) + 1$	0
$4 - 4\sqrt{3} + 3 - 8 + 4\sqrt{3} + 1$	0
$0 + 0$	0
0	0
LS =	RS

The two roots are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

If $x^2 = 3$, then $x = \pm\sqrt{3}$.

If $(x-2)^2 = 3$, then

$$x-2 = \pm\sqrt{3}.$$

If $x = \pm\sqrt{3}$, then $x = \sqrt{3}$ or $x = -\sqrt{3}$.

If $x = 2 \pm \sqrt{3}$, then $x = 2 + \sqrt{3}$ or $x = 2 - \sqrt{3}$.

The $\pm$ implies that there are two roots.

From the graph in Example 1 you approximated the roots of $x^2 - 4x + 1 = 0$ to be 0.25 and 3.75. Check to see how close these approximations are to the exact values.

The exact values are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

Since $\sqrt{3} \doteq 1.73$, (correct to two decimal places) then the values are as follows:

$$2 + \sqrt{3} \doteq 2 + 1.73 \\ \doteq 3.73$$

$$2 - \sqrt{3} \doteq 2 - 1.73 \\ \doteq 0.27$$

As you can see, the approximate values obtained from the graph are reasonably close to the exact values.

The next example is a little more difficult since the equation contains a numerical coefficient in front of the x^2 .

Example 3

Find the roots of the quadratic equation $3x^2 + 2x - 4 = 0$ by completing the square.

Solution:

$$3x^2 + 2x - 4 = 0$$

Put parentheses around the terms containing x^2 and x .

$$(3x^2 + 2x) - 4 = 0$$

Move the coefficient of x^2 from inside the parentheses.

$$3\left(x^2 + \frac{2}{3}x\right) - 4 = 0$$

Complete the square inside the parentheses and subtract the appropriate value outside the parentheses.

$$\begin{aligned} \left(\frac{1}{2} \times \frac{2}{3}\right)^2 &= \left(\frac{1}{3}\right)^2 = \frac{1}{9} \\ 3\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) - 4 - 3\left(\frac{1}{9}\right) &= 0 \end{aligned}$$

Note: The symbol $\doteq$ means approximately equal to.

To factor 3 from $3x^2 + 2x$, you must divide $3x^2 + 2x$ by 3.

$$\frac{3x^2}{3} + \frac{2x}{3} = x^2 + \frac{2}{3}x$$

Remember to write 3 in front of the parentheses.

Change the trinomial inside the parentheses to a perfect square.

$$3\left(x + \frac{1}{3}\right)^2 - 4 - \frac{1}{3} = 0$$

Simplify. The equation is in completed square form.

$$3\left(x + \frac{1}{3}\right)^2 - \frac{13}{3} = 0$$

Now solve for x .

Isolate the binomial and eliminate the 3 on the left side by multiplying both sides by $\frac{1}{3}$.

$$3\left(x + \frac{1}{3}\right)^2 = \frac{13}{3}$$

$$\left(\frac{1}{3}\right)\left(x + \frac{1}{3}\right)^2 = \left(\frac{1}{3}\right)\left(\frac{13}{3}\right)$$

$$\left(x + \frac{1}{3}\right)^2 = \frac{13}{9}$$

$$x + \frac{1}{3} = \pm\sqrt{\frac{13}{9}}$$

$$x + \frac{1}{3} = \pm\frac{\sqrt{13}}{3}$$

(Take the square root of both sides.)

(Note that $\sqrt{\frac{13}{9}} = \frac{\sqrt{13}}{\sqrt{9}} = \frac{\sqrt{13}}{3}$.)

$$x = -\frac{1}{3} \pm \frac{\sqrt{13}}{3}$$

(Isolate the x .)

$$x = \frac{-1 \pm \sqrt{13}}{3}$$

(The $\pm$ indicates that there are two roots.)

$$x = \frac{-1 + \sqrt{13}}{3} \quad \text{or} \quad x = \frac{-1 - \sqrt{13}}{3}$$

The two roots are $\frac{-1 + \sqrt{13}}{3}$ and $\frac{-1 - \sqrt{13}}{3}$.

The approximate decimal values can also be calculated for the roots obtained in Example 3.

Since $\sqrt{13} \doteq 3.61$, the values are as follows:

$$\frac{-1 + \sqrt{13}}{3} \doteq \frac{-1 + 3.61}{3}$$

$$\doteq \frac{2.61}{3}$$

$$\doteq 0.87$$

$$\frac{-1 - \sqrt{13}}{3} \doteq \frac{-1 - 3.61}{3}$$

$$\doteq \frac{-4.61}{3}$$

$$\doteq -1.54$$

The graph of the corresponding function would have x -intercepts of approximately 0.87 and -1.54 .

The roots for any quadratic equation (even ones that are factorable) can be found by completing the square. This method can be applied to the standard form of the quadratic equation or $ax^2 + bx + c = 0$. The result is a formula that can be used to solve for the roots of any quadratic equation. To complete the square for $ax^2 + bx + c = 0$, you would proceed as follows.

$$ax^2 + bx + c = 0$$

$$(ax^2 + bx) + c = 0$$

$$a\left(x^2 + \frac{b}{a}x\right) + c = 0$$

Write the equation in standard form.

Put parentheses around the terms containing x^2 and x .

Move a from inside the parentheses.

$$\left(\frac{1}{2} \times \frac{b}{a}\right)^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right) = 0$$

Complete the square inside the parentheses and subtract the appropriate value outside the parentheses.

$$a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = 0$$

Change the trinomial inside the parentheses to a perfect square.

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac}{4a} - \frac{b^2}{4a} = 0$$

Simplify.

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0$$

Simplify.

$$\begin{aligned} c - \frac{b^2}{4a} &= \frac{c(4a)}{4a} - \frac{b^2}{4a} \\ &= \frac{4ac - b^2}{4a} \\ &= \frac{4ac - b^2}{4a} \end{aligned}$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a}$$

Isolate the binomial.

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$$

$$\left(\frac{1}{a}\right)^1 \left(\frac{1}{a}\right)^1 \left(x + \frac{b}{2a}\right)^2 = \left(\frac{1}{a}\right)^1 \left(\frac{b^2 - 4ac}{4a}\right)$$

Multiply both sides by $\frac{1}{a}$.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Take the square root of both sides.

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Isolate the x .

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This expression is called the **quadratic formula**. It can be used to find the roots for any quadratic equation provided the values of a , b , and c are known.

$$\frac{-4ac + b^2}{4a} = \frac{b^2 - 4ac}{4a}$$

$$\sqrt{4a^2} = 2a$$

Note: The quadratic formula should be **memorized**.



The quadratic formula

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ can be used to find the roots of any quadratic equation in the form

$ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

The formula tells you that there are two roots for any quadratic equation. The two roots are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$



The quadratic formula allows you to calculate the roots of any quadratic equation by using the values of a, b , and c . Because of this formula, the long method of completing the square does not have to be used. Study the next example to see how this formula is applied.

Example 4

Find the roots for $2x^2 + 8x + 3 = 0$.

Solution:

Step 1: Make sure the equation is in standard form.

Step 2: Identify the values of a, b , and c .

$$\begin{array}{ccc} 2x^2 + 8x + 3 = 0 & & \\ \downarrow & \searrow & \\ a = 2 & b = 8 & c = 3 \end{array}$$

Step 3: Substitute the values of a, b , and c into the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$x = \frac{-8 \pm \sqrt{(8)^2 - 4(2)(3)}}{2(2)}$$

$$x = \frac{-8 \pm \sqrt{64 - 24}}{4}$$

$$x = \frac{-8 \pm \sqrt{40}}{4}$$

$$x = \frac{-8 \pm 2\sqrt{10}}{4}$$

Factor 2 from the top and the bottom. These can be eliminated by cancelling.

$$x = \frac{2(-4 \pm \sqrt{10})}{2(2)}$$

$$x = \frac{-4 \pm \sqrt{10}}{2}$$

(The $\pm$ indicates that there are two roots.)

The roots are $\frac{-4 + \sqrt{10}}{2}$ and $\frac{-4 - \sqrt{10}}{2}$. These roots are exact.

In $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, a is the numerical coefficient of x^2 , b is the numerical coefficient of x , and c is the constant term at the end in $ax^2 + bx + c = 0$.

Standard form for a quadratic equation is as follows:

$ax^2 + bx + c = 0$

The x^2 term is first.

The x -term is second.

The constant term is last.

$$\left(\sqrt{40} = \sqrt{4 \times \sqrt{10}} \right) = 2\sqrt{10}$$

The equation in Example 4 has two roots; therefore, the graph of the corresponding function must have two x -intercepts. The approximate value of these roots can be found by letting $\sqrt{10} \doteq 3.16$.

$$\begin{aligned}\frac{-4 + \sqrt{10}}{2} &\doteq \frac{-4 + 3.16}{2} \\ &\doteq \frac{-0.84}{2} \\ &\doteq -0.42\end{aligned}$$

$$\begin{aligned}\frac{-4 - \sqrt{10}}{2} &\doteq \frac{-4 - 3.16}{2} \\ &\doteq \frac{-7.16}{2} \\ &\doteq -3.58\end{aligned}$$

These roots are approximate.



Example 5

Solve the equation $4x^2 = 12x - 9$.

Solution:

Step 1: $4x^2 - 12x + 9 = 0$

Step 2: $a = 4$, $b = -12$, $c = 9$

Do not forget the negative sign for $b = -12$.

$$\begin{aligned}\text{Step 3: } x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)} \\ x &= \frac{12 \pm \sqrt{144 - 144}}{8} \\ x &= \frac{12 \pm \sqrt{0}}{8} \\ x &= \frac{12 \pm 0}{8} \\ x &= \frac{12}{8} \\ x &= \frac{3}{2}\end{aligned}$$

The root of the equation is $\frac{3}{2}$ or $1\frac{1}{2}$.

To solve a quadratic equation means to find the roots of the equation.

This equation actually has two roots, but they are equal. The two roots are as follows:

$$\frac{12 + 0}{8} = \frac{12}{8} = \frac{3}{2}$$

$$\frac{12 - 0}{8} = \frac{12}{8} = \frac{3}{2}$$

If the roots are equal, just give that value once.

The equation in Example 5 has one root; therefore, the graph of the corresponding function will intersect the x -axis at one point. That point is $\frac{3}{2}$.

The equation in Example 5 can also be solved by factoring.

$$\begin{aligned} 4x^2 - 12x + 9 &= 0 \\ (2x - 3)(2x - 3) &= 0 \\ 2x - 3 &= 0 \quad \text{or} \quad 2x - 3 = 0 \\ 2x &= 3 & 2x &= 3 \\ x &= \frac{3}{2} & x &= \frac{3}{2} \end{aligned}$$

The root is $\frac{3}{2}$.

Which method do you find easier?

To find the roots for a quadratic equation, you should always try to factor first. If you are unable to factor, then use the quadratic formula.



The next equation does not have any real roots.

Example 6

Find the roots of $-2x^2 + 5x - 6 = 6$.

Solution:

Step 1: $-2x^2 + 5x - 6 = 0$

Step 2: $a = -2, b = 5, c = -6$

Step 3: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4(-2)(-6)}}{2(-2)}$$

$$x = \frac{-5 \pm \sqrt{25 - 48}}{-4}$$

$$x = \frac{-5 \pm \sqrt{-23}}{-4}$$

You can tell from Step 3 that the equation will not have any real roots because $\sqrt{-23}$ is not a real number.

The equation does not have any real roots; therefore, the graph of the corresponding function will not have any x -intercepts.

Make sure that the values of a , b , and c are correct.

$$\begin{array}{l} -2x^2 + 5x - 6 = 0 \\ \swarrow \quad \searrow \quad \nearrow \\ a = -2 \quad b = 5 \quad c = -6 \end{array}$$

Do not forget the negative signs.

Note that $\sqrt{-23}$ is not a real number. It is an imaginary number.

Example 7

Find the roots for each quadratic equation that follows.

$$\bullet 5x^2 - 4x = 0$$

Solution:

$$5x^2 - 4x = 0$$

$$a = 5, b = -4, c = 0$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(5)(0)}}{2(5)}$$

$$x = \frac{4 \pm \sqrt{16 - 0}}{10}$$

$$x = \frac{4 \pm 4}{10}$$

$$x = \frac{4+4}{10} \text{ or } x = \frac{4-4}{10}$$

$$x = \frac{8}{10} \qquad x = \frac{0}{10}$$

$$x = \frac{4}{5} \qquad x = 0$$

The roots are $\frac{4}{5}$ and 0.

$$\bullet 3x^2 - 7 = 0$$

Solution:

$$3x^2 - 7 = 0$$

$$a = 3, b = 0, c = -7$$

$$x = \frac{-0 \pm \sqrt{(0)^2 - 4(3)(-7)}}{2(3)}$$

$$x = \frac{0 \pm \sqrt{0 + 84}}{6}$$

$$x = \frac{\pm \sqrt{84}}{6}$$

$$x = \frac{\pm 2\sqrt{21}}{6}$$

$$x = \pm \frac{\sqrt{21}}{3}$$

$$x = \frac{\sqrt{21}}{3} \text{ or } x = -\frac{\sqrt{21}}{3}$$

The roots are $\frac{\sqrt{21}}{3}$ and $-\frac{\sqrt{21}}{3}$.

$$\begin{aligned}\sqrt{84} &= \sqrt{4 \times \sqrt{21}} \\ &= 2\sqrt{21}\end{aligned}$$

The two equations in Example 7 also could have been solved by using previously taught methods such as the following.

$$\begin{aligned} & \bullet \quad 5x^2 - 4x = 0 \\ & \quad x(5x - 4) = 0 \\ & \quad x = 0 \quad \text{or} \quad 5x - 4 = 0 \\ & \quad \quad \quad 5x = 4 \\ & \quad \quad \quad x = \frac{4}{5} \end{aligned}$$

The roots are 0 and $\frac{4}{5}$.

$$\bullet \quad 3x^2 - 7 = 0$$

$$3x^2 = 7$$

$$x^2 = \frac{7}{3}$$

$$\begin{aligned} x &= \pm \sqrt{\frac{7}{3}} \\ &= \pm \frac{\sqrt{21}}{3} \end{aligned}$$

The roots are $\frac{\sqrt{21}}{3}$ and $-\frac{\sqrt{21}}{3}$.

As you can see, it is much easier to find the roots using methods taught previously. Only use the quadratic formula when other methods will not work.

Now that you have the necessary skills to solve quadratic equations, it is time for you to use these skills to solve problems. Follow the next example.

Example 8

An emergency flare is projected upward with a velocity of 80 m/s. The function that describes the trajectory of the flare is $h = -5t^2 + 80t$, where h is the height above ground level in metres and t is the time in seconds.

- Find the time it takes the flare to reach a height of 300 m.

Solution:

Substitute 300 for h in $h = -5t^2 + 80t$.

$$300 = -5t^2 + 80t$$

Collect all the terms on one side to get a quadratic equation.

$$5t^2 - 80t + 300 = 0$$

$$\begin{aligned} \sqrt{\frac{7}{3}} &= \frac{\sqrt{7}}{\sqrt{3}} \\ &= \frac{\sqrt{7} \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} \\ &= \frac{\sqrt{21}}{3} \end{aligned}$$

You could also collect all the terms on the right side. The resulting equation

$$\begin{aligned} -5t^2 + 80t - 300 &= 0 \quad \text{will have} \\ &\text{the same roots as} \\ 5t^2 - 80t + 300 &= 0. \end{aligned}$$

Simplify the quadratic equation by dividing both sides by 5.

$$t^2 - 16t + 60 = 0$$

Factor the quadratic equation.

$$(t-6)(t-10) = 0$$

$$t-6 = 0 \quad \text{or} \quad t-10 = 0$$

$$t = 6 \qquad t = 10$$

The flare will be 300 m above ground level after 6 s (when the flare is on its way up) and after 10 s (when the flare is on its way down).

- Find the time it takes the flare to reach a height of 225 m.

Solution:

Substitute 225 for h in $h = -5t^2 + 80t$.

$$225 = -5t^2 + 80t$$

Collect all terms on one side to get a quadratic equation.

$$5t^2 - 80t + 225 = 0$$

Simplify the equation by dividing both sides by 5.

$$t^2 - 16t + 45 = 0$$

This equation is not readily factorable. Therefore, to solve it, you should use the quadratic formula.

$$t^2 - 16t + 45 = 0$$

$$a = 1, b = -16, c = 45$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(1)(45)}}{2(1)}$$

$$t = \frac{16 \pm \sqrt{256 - 180}}{2}$$

$$t = \frac{16 \pm \sqrt{76}}{2}$$

$$t = \frac{16 \pm 2\sqrt{19}}{2}$$

$$\frac{5t^2 - 80t + 225}{5} = \frac{0}{5}$$

$$\frac{5t^2 - 80t + 225}{5} = \frac{0}{5}$$

$$t^2 - 16t + 45 = 0$$

If a quadratic equation contains a variable other than x , the quadratic formula is expressed in terms of that variable. For example, if $t^2 - 16t + 45 = 0$, then $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$\left(\begin{aligned} \sqrt{76} &= \sqrt{4 \times \sqrt{19}} \\ &= 2\sqrt{19} \end{aligned} \right)$$

Factor 2 from the top and cancel it out with the 2 on the bottom.

$$t = \frac{\cancel{2}(8 \pm \sqrt{19})}{\cancel{2}}$$

$$t = 8 \pm \sqrt{19}$$

The flare will be 225 m above ground after $(8 + \sqrt{19})$ seconds and $(8 - \sqrt{19})$ seconds. These roots are exact.

Since $\sqrt{19} \doteq 4.4$, the values may be expressed as decimal numbers.

$$8 + \sqrt{19} \doteq 8 + 4.4 \\ \doteq 12.4$$

$$8 - \sqrt{19} \doteq 8 - 4.4 \\ \doteq 3.6$$

These roots are approximate.

The flare will be 225 m above ground after about 3.6 s (on its way up) and after about 12.4 s (on its way down).

• Find the time it takes for the flare to strike the ground.

Solution:

When the flare strikes the ground the height above ground level will be zero ($h = 0$). Therefore, substitute 0 for h .

$$0 = -5t^2 + 80t$$

Divide both sides by -5 (or 5) to simplify the equation.

$$\frac{0}{-5} = \frac{-5t^2 + 80t}{-5}$$

$$0 = t^2 - 16t$$

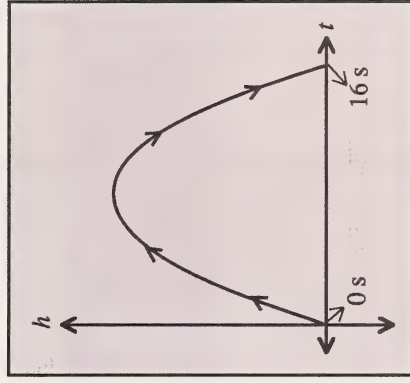
Factor.

$$0 = t(t - 16)$$

$$t = 0 \text{ or } t - 16 = 0$$

$$t = 0 \qquad t = 16$$

The flare will strike the ground after 16 s. (When $t = 0$, the flare has not been projected.)



- Find the height of the flare above ground level after 5 s.

Solution:

To find the height after five seconds, substitute $t = 5$ in the function

$$h = -5t^2 + 80t.$$

$$h = -5(5)^2 + 80(5)$$

$$h = -5(25) + 400$$

$$h = -125 + 400$$

$$h = 275$$

After five seconds, the flare will be 275 m above ground level.

In many cases you will have to set up your own equation and then solve it. Follow the next example.

Example 9

Find two numbers such that their sum is 17 and their product is 52.

Solution:

Since the two numbers are unknown, you will have to define them in terms of a variable.

Let x be one of the numbers.

Since the sum is 17, then $17 - x$ must be the other number.

If the two numbers have a product of 52, then you can write $x(17 - x) = 52$.

$$x(17 - x) = 52$$

$$17x - x^2 = 52$$

Put this in standard form, and eliminate the negative in front of the x^2 by dividing both sides by -1 .

$$-x^2 + 17x - 52 = 0$$

$$x^2 - 17x + 52 = 0$$

Factor the quadratic equation.

$$(x - 4)(x - 13) = 0$$

$$x - 4 = 0 \quad \text{or} \quad x - 13 = 0$$

$$x = 4 \qquad x = 13$$

(There are two possibilities.)

When one number is 4, the other number is

$$17 - 4 = 13.$$

When one number is 13, the other number is

$$17 - 13 = 4.$$

The two numbers are 4 and 13.

The two numbers are x and $17 - x$.

$$\frac{-x^2 + 17x - 52}{-1} = \frac{0}{-1}$$

$$x^2 - 17x + 52 = 0$$

If a problem can be visualized with a diagram, then a quick sketch of the situation may help you solve the problem.

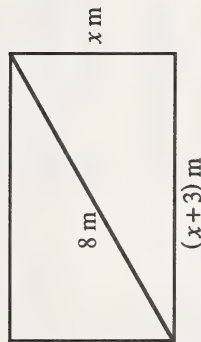
Example 10

The floor in a warehouse is rectangular and has a diagonal length of 8 m. Find the dimensions of the floor if the length is 3 m more than the width. Round your answers to two digits after the decimal.

Solution:

The unknowns are the width and the length of the floor. Define these unknowns in terms of a variable.

Let x be the width of the floor; therefore, $x + 3$ has to be the length.



The width, length, and diagonal form a right-angled triangle. Use the Pythagorean theorem to solve the triangle.

$$(8)^2 = (x)^2 + (x + 3)^2$$

$$64 = x^2 + x^2 + 6x + 9$$

$$64 = 2x^2 + 6x + 9$$

$$0 = 2x^2 + 6x - 55$$

This equation is not factorable, so solve it by using the quadratic formula.

$$0 = 2x^2 + 6x - 55$$

$$a = 2, b = 6, c = -55$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-6 \pm \sqrt{(6)^2 - 4(2)(-55)}}{2(2)}$$

$$x = \frac{-6 \pm \sqrt{36 + 440}}{4}$$

$$x = \frac{-6 \pm \sqrt{476}}{4}$$



$$\sqrt{476} \div 21.81742423$$

$$x \div \frac{-6 + 21.81742423}{4}$$

or

$$x \div 3.954356057$$

$$x \div 3.95 \text{ (rounded)}$$

Since x represents the width, the width is about 3.95 m.

The length is $(x + 3)$ or about 3.95 m + 3 m = 6.95 m.

The dimensions are about 3.95 m by 6.95 m.

The next example involves three unknowns.

$$x \div \frac{-6 - 21.81742423}{4}$$

$$x \div -6.954356057$$

Since x represents the width, this root is **inadmissible** because the width of the floor cannot be a negative quantity.

Example 11

The sum of the squares of three consecutive integers is 302. What are the integers?

Solution:

Let n be the first integer, $n + 1$ be the second integer, and $n + 2$ be the third integer.

If the sum of the squares of these integers is 302, then you can write the following:

$$n^2 + (n + 1)^2 + (n + 2)^2 = 302$$

$$n^2 + n^2 + 2n + 1 + n^2 + 4n + 4 = 302$$

$$3n^2 + 6n + 5 = 302$$

$$3n^2 + 6n - 297 = 0$$

$$n^2 + 2n - 99 = 0 \quad (\text{Divide both sides by 3.})$$

$$(n + 11)(n - 9) = 0 \quad (\text{Factor the quadratic equation.})$$

$$n + 11 = 0 \quad \text{or} \quad n - 9 = 0$$

$$n = -11 \quad n = 9$$

When $n = -11$, the integers are -11 , $(-11 + 1)$, and $(-11 + 2)$ or -11 , -10 , and -9 .

When $n = 9$, the integers are 9 , $(9 + 1)$, and $(9 + 2)$ or 9 , 10 , and 11 .

There are two possible sets of answers for this problem.

The integers are 9 , 10 , and 11 or -9 , -10 , and -11 .

It is time for you to test your understanding of the concepts in this topic. Do the following exercises.

- Put each quadratic equation that follows in standard form and identify the values of a , b , and c .

a. $3 + 2x^2 - 3x = 0$

b. $-7 - 2x^2 = 0$

c. $-4x + 3x^2 = 0$

- Solve for x in the following equations. Do any five.

a. $x^2 = 9$

b. $x^2 = 5$

c. $x^2 = -4$

d. $(x-2)^2 = 9$

e. $(x+2)^2 = 25$

f. $(x-3)^2 = 7$

g. $(2x+3)^2 = 4$

h. $(3x-7)^2 = 5$

- Verify that $2 + \sqrt{2}$ and $2 - \sqrt{2}$ are the roots of the equation $x^2 - 4x + 2 = 0$.

- What are the approximate values of the roots $\frac{5+\sqrt{7}}{2}$ and $\frac{5-\sqrt{7}}{2}$ rounded to one decimal place?

- Factor 3 from each binomial that follows.

a. $3x^2 - 6x$

b. $3x^2 - 2x$

c. $3x^2 + 5x$

- What is the quadratic formula and when should it be used?

- Find the roots for the following quadratic equations. Do any five.

a. $x^2 - 7x + 6 = 0$

b. $-2x^2 - 5x + 4 = 0$

c. $x^2 - 144 = 0$

d. $3x^2 - 2x = 3$

e. $6 - 4x + x^2 = 0$

f. $3x^2 - 12x = 0$

g. $16x^2 - 2x + 1 = 0$

h. $0 = x^2 - x - 1$

i. $x^2 - \frac{1}{2}x - \frac{3}{4} = 0$

- Find the x -intercepts for the graph of the function

$y = 5x^2 - 4x - 3$. Express your answer as a decimal number rounded to two digits after the decimal.

9. The trajectory of a projectile is defined by the function $h = -5t^2 + 100t + 20$, where h is the height in metres and t is the time in seconds. Calculate the following.
 - a. the time when the height is 340 m
 - b. the time when the height is 250 m
 - c. the height when the time is $2\sqrt{3}$ s
10. Do any three of the following problems.
 - a. The area of a rectangular field is 2100 m^2 . Find the dimensions of this field if the length is 40 m more than the width.
 - b. How wide should a uniform border be on a page 9 cm by 11 cm such that 48 cm^2 is available for printing?
 - c. Find two consecutive positive integers such that the sum of their squares is 421.

- d. Find two consecutive positive even integers such that the sum of their squares is 452.
- e. One side of a right triangle is 7 cm longer than the other side. The hypotenuse is 17 cm. Find the length of each side of the triangle.



For solutions to **Activity 1**, turn to **Appendix A, Topic 5**.



Use Lesson 38 of the computer program titled *Problem Solving in Algebra*¹ for additional problems involving quadratics. Do as many questions as you feel are necessary.



¹ *Problem Solving in Algebra* is a title of Encyclopedia Britannica Publications, Ltd.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.



Extra Help

When solving quadratic equations, you should try to solve the equation by factoring. Factoring is the easiest and quickest way to solve a quadratic equation.

If a quadratic equation is not readily factorable, you can solve it using the quadratic formula. The quadratic formula may be used to solve any quadratic equation, including the ones that are factorable. To give you more practice in solving quadratic equations, do the following exercises.

Find the roots for each quadratic equation that follows by using the quadratic formula.

1. $x^2 - 5x + 2 = 0$

2. $x^2 + 3x - 7 = 0$

3. $3x^2 + 5x + 1 = 0$

4. $2x^2 - 3x - 1 = 0$

5. $6x^2 - 2x + 1 = 0$

6. $-4x^2 - 7x - 3 = 0$



For solutions to **Extra Help**, turn to **Appendix A, Topic 5**.





Extensions

All quadratic equations in this topic have had rational number coefficients. It is possible to solve quadratic equations with irrational number coefficients. For example, the equation $\sqrt{3}x^2 - 4\sqrt{2}x - 3\sqrt{5} = 0$ has irrational number coefficients. Look at the following example to see how you would solve an equation like this.

Example 12

Find the roots of $\sqrt{3}x^2 - 4\sqrt{2}x - 3\sqrt{5} = 0$. Also give an approximate value for these roots as a decimal number rounded to two digits after the decimal point.

Solution:

$$\sqrt{3}x^2 - 4\sqrt{2}x - 3\sqrt{5} = 0$$

This equation is not readily factorable; therefore, use the quadratic formula. Identify the values of a , b , and c .

$$a = \sqrt{3}, b = -4\sqrt{2}, c = -3\sqrt{5}$$

Put the values into the quadratic formula.

$$x = \frac{-(-4\sqrt{2}) \pm \sqrt{(-4\sqrt{2})^2 - 4(\sqrt{3})(-3\sqrt{5})}}{2(\sqrt{3})}$$

$$x = \frac{4\sqrt{2} \pm \sqrt{32 + 12\sqrt{15}}}{2\sqrt{3}}$$

The roots are $x = \frac{4\sqrt{2} + \sqrt{32 + 12\sqrt{15}}}{2\sqrt{3}}$ and

$$x = \frac{4\sqrt{2} - \sqrt{32 + 12\sqrt{15}}}{2\sqrt{3}}.$$



$$\begin{aligned} (-4\sqrt{2})^2 &= (-4\sqrt{2})(-4\sqrt{2}) \\ &= 16\sqrt{4} \\ &= 16(2) \\ &= 32 \end{aligned}$$

Note that $\sqrt{3}$, $-4\sqrt{2}$, and $-3\sqrt{5}$ are all irrational numbers.



These roots are the exact roots. To find the approximate decimal values of these roots, use your calculator.

$$\frac{4\sqrt{2} + \sqrt{32 + 12\sqrt{15}}}{2\sqrt{3}} \div \frac{4(1.414213562) + \sqrt{32 + 12(3.872983346)}}{2(1.732050808)}$$

$$\div \frac{5.656854249 + \sqrt{32 + 46.47580015}}{3.46101615}$$

$$\div \frac{5.656854249 + \sqrt{78.47580015}}{3.46101615}$$

$$\div \frac{5.656854249 + 8.858656792}{3.464101615}$$

$$\div \frac{14.51551104}{3.464101615}$$

$$\div 4.190267103$$

$$\div 4.19$$

$$\frac{4\sqrt{2} - \sqrt{32 + 12\sqrt{15}}}{2\sqrt{3}} \div \frac{4(1.414213562) - \sqrt{32 + 12(3.872983346)}}{2(1.732050808)}$$

$$\div \frac{5.656854249 - \sqrt{32 + 46.47580015}}{3.464101615}$$

$$\div \frac{5.656854249 - \sqrt{78.47580015}}{3.464101615}$$

$$\div \frac{5.656854249 - 8.858656792}{3.464101615}$$

$$\div \frac{-3.201802543}{3.464101615}$$

$$\div -0.92428078$$

$$\div -0.92$$

The approximate values of the roots rounded to two decimal place are 4.19 and -0.92.

Try the following problem.

Find the roots of the quadratic equation $2\sqrt{3}x^2 - \sqrt{2}x - 3\sqrt{7} = 0$. Also give a decimal approximation for each root rounded to two digits after the decimal point.



For solutions to Extensions, turn to **Appendix A, Topic 5**.

Topic 6 Extensions of Quadratic Equations



Introduction

In Unit 1 and Unit 2 of this course you learned how to solve radical equations and equations involving rational expressions. In this topic you will use your skills in solving quadratic equations to solve radical equations and equations with more complex rational expressions.



What Lies Ahead

Throughout the topic you will learn to

1. solve radical equations and equations involving rational expressions both of which transform into quadratic equations

Now that you know what to expect, turn the page to begin your study of extensions of quadratic equations.



Exploring Topic 6

Activity 1



Solve radical equations and equations involving rational expressions both of which transform into quadratic equations.

In Unit 1 of this course you learned how to solve radical equations. Recall that a radical equation is any equation that contains a variable under a radical sign. Examples of radical equations are $\sqrt{x+3}=7$, $\sqrt{2x-3}=5$, and $3\sqrt{x-2}=2\sqrt{3x-7}$. Each of these equations has a variable under a radical sign.

Most radical equations can be solved by using the methods outlined in Unit 1. However, some radical equations, in the process of being solved, transform into quadratic equations. These quadratic equations are solved using the methods learned in this unit. Look at the first example which will refresh your memory about solving radical equations.

Example 1

Solve the following radical equations.

$$\bullet \sqrt{x+3}=7$$

Solution:

$$\sqrt{x+3}=7$$

$$\sqrt{x+3}-3=7-3 \quad (\text{Isolate the radical.})$$

$$\sqrt{x}=4 \quad (\text{Simplify.})$$

$$(\sqrt{x})^2=(4)^2 \quad (\text{Square both sides.})$$

$$x=16$$

Recall that all possible solutions for a radical equation must be checked. This is necessary because when both sides of an equation are squared, extraneous roots may be introduced. An extraneous root is a root that does not satisfy the original equation. To determine whether 16 is a root, you must replace the x with 16 in the original equation and see if $LS=RS$.

To solve a radical equation, you isolate the radical and then square both sides to eliminate the radical sign. Then solve for the variable.

Check:

$$x = 16$$

LS	RS
$\sqrt{x} + 3$	7
$\sqrt{16} + 3$	7
$4 + 3$	7
7	7
LS	RS

(Use the original equation.)

(Substitute $x = 16$.)

Therefore, $x = 16$ is the solution.

$$\bullet 2\sqrt{3x-7} + 5 = 11$$

Solution:

$$2\sqrt{3x-7} + 5 = 11$$

(Isolate the radical.)

$$2\sqrt{3x-7} + 5 - 5 = 11 - 5$$

(Eliminate the 5 from the LS.)

$$2\sqrt{3x-7} = 6$$

$$\frac{2\sqrt{3x-7}}{2} = \frac{6}{2}$$

(Divide both sides by 2.)

$$\sqrt{3x-7} = 3$$

$$(\sqrt{3x-7})^2 = (3)^2$$

(Square both sides of the equation.)

$$3x - 7 = 9$$

$$3x - 7 + 7 = 9 + 7$$

$$3x = 16$$

$$\frac{3x}{3} = \frac{16}{3}$$

$$x = \frac{16}{3}$$

(Isolate the variable by eliminating the 7 from the LS.)

(Divide both sides by 3.)

Check:

$$x = \frac{16}{3}$$

LS	RS
$2\sqrt{3x-7} + 5$	11
$2\sqrt{3\left(\frac{16}{3}\right) - 7} + 5$	11
$2\sqrt{16-7} + 5$	11
$2\sqrt{9} + 5$	11
$2(3) + 5$	11
$6 + 5$	11
11	11
LS	RS

(Use the original equation.)

(Substitute $x = \frac{16}{3}$.)

Therefore, $x = \frac{16}{3}$ is a root.

As stated earlier, some radical equations transform or simplify into quadratic equations. The next example shows you how to solve these equations.

Example 2

Solve the equation $\sqrt{x+22} - 2 = x$.

Solution:

Isolate the radical.

$$\sqrt{x+22} - 2 = x$$

$$\sqrt{x+22} = x+2$$

Square both sides of the equation.

$$(\sqrt{x+22})^2 = (x+2)^2$$

$$x+22 = x^2 + 4x + 4$$

Collect all the terms on one side. Note that this is a quadratic equation.

$$0 = x^2 + 3x - 18$$

Factor the quadratic equation.

$$0 = (x+6)(x-3)$$

$$x+6=0 \text{ or } x-3=0$$

$$x = -6 \quad x = 3$$

Before these roots can be accepted as solutions to the equation, they must be verified.

Check:

$$x = -6$$

LS	RS
$\sqrt{x+22} - 2$	x
$\sqrt{-6+22} - 2$	-6
$\sqrt{16} - 2$	-6
$4 - 2$	-6
2	-6
LS	$\neq$ RS

Therefore, $x = -6$ is not a root. Thus, $x = -6$ is an extraneous root.

Check:

$$x = 3$$

LS	RS
$\sqrt{x+22} - 2$	x
$\sqrt{3+22} - 2$	3
$\sqrt{25} - 2$	3
$5 - 2$	3
3	3
LS	$=$ RS

Therefore, $x = 3$ is a root.

The solution to the equation is $x = 3$.

To verify the roots for some radical equations, you need to use your calculator. Look at the next example.

Example 3

Find the roots of the equation $3\sqrt{x} + 5 = x + 3$.

Solution:

Isolate the radical.

$$3\sqrt{x} + 5 = x + 3$$

$$3\sqrt{x} = x - 2$$

Square both sides.

$$(3\sqrt{x})^2 = (x - 2)^2$$

$$9x = x^2 - 4x + 4$$

Collect all the terms on one side. This equation is a quadratic equation.

$$0 = x^2 - 13x + 4$$

It is not factorable, so you need to use the quadratic formula.

$$a = 1, b = -13, c = 4$$

$$x = \frac{-(-13) \pm \sqrt{(-13)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{13 \pm \sqrt{169 - 16}}{2}$$

$$x = \frac{13 \pm \sqrt{153}}{2}$$

To verify that $x = \frac{13 + \sqrt{153}}{2}$ and $x = \frac{13 - \sqrt{153}}{2}$ are the roots of the equation, you will find it easier to use a decimal approximation for these roots.



$$\text{Check: } x = \frac{13 + \sqrt{153}}{2}$$

$$\sqrt{153} \div 12.36931688$$

$$x = \frac{13 + 12.36931688}{2}$$

$$x \div 12.68465844$$

LS	RS
$3\sqrt{x} + 5$	$x + 3$
$3\sqrt{12.68465844} + 5$	$12.68465844 + 3$
$3(3.561552813) + 5$	15.68465844
$10.68465844 + 5$	15.68465844
15.68465844	15.68465844
LS	RS

Therefore, $x = \frac{13 + \sqrt{153}}{2}$ or $x = \frac{13 + 3\sqrt{17}}{2}$ is a root for the equation.

Remember that $\sqrt{x}$ is a radical, and $3\sqrt{x}$ is a mixed radical.

When verifying a root, use as many digits after the decimal point as possible. Use the value in your calculator display.

$$\begin{aligned}\sqrt{153} &= \sqrt{9 \times \sqrt{17}} \\ &= 3\sqrt{17}\end{aligned}$$

Check: $x = \frac{13 - \sqrt{153}}{2}$

$$\sqrt{153} \div 12.36931688$$

$$x \div \frac{13 - 12.36931688}{2}$$

$$x \div \frac{0.63068312}{2}$$

$$x \div 0.31534156$$

LS	RS
$3\sqrt{x+5}$	$x+3$
$3\sqrt{0.31534156+5}$	$0.31534156+3$
$3(0.561552811+5)$	3.31534156
$1.684658434+5$	3.31534156
6.684658434	3.31534156
LS	$\neq$ RS

Therefore, $x = \frac{13 - \sqrt{153}}{2}$ is not a root of the equation.

You have seen how some radical equations can be changed to quadratic equations. The same thing happens to some equations that involve rational expressions. Quite often in the process of solving an equation with rational expressions you will be required to solve a quadratic equation.

The next example shows you how to solve an equation with rational expressions.

Example 4

Solve each equation that follows.

$$\bullet \frac{x-1}{2} + \frac{2x+3}{4} = 3$$

Solution:

Identify the L.C.D. of all the denominators.
The L.C.D. of 2 and 4 is 4.

$$\frac{x-1}{2} + \frac{2x+3}{4} = 3$$

Multiply both sides of the equation by the L.C.D. to eliminate the denominators.

$$4\left(\frac{x-1}{2} + \frac{2x+3}{4}\right) = 4(3)$$

Eliminate the denominators by reducing wherever possible.

$$4\left(\frac{x-1}{2}\right) + 4\left(\frac{2x+3}{4}\right) = 12$$

$$2(x-1) + (2x+3) = 12$$

L.C.D. means lowest common denominator.

The purpose of multiplying both sides by the L.C.D. is to eliminate the denominator.

Expand and solve for x .

$$2x - 2 + 2x + 3 = 12$$

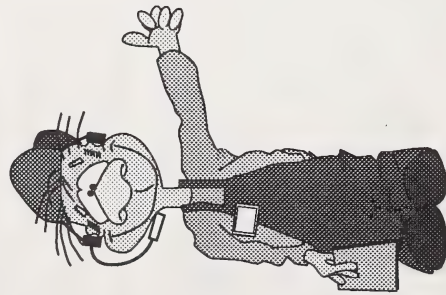
$$4x + 1 = 12$$

$$4x = 11$$

$$x = \frac{11}{4}$$

The root is $\frac{11}{4}$.

The solutions to equations that involve rational expressions do not have to be verified, but for interest, replace x with $\frac{11}{4}$ to see if $\frac{11}{4}$ does satisfy the equation.



Check:

$$x = \frac{11}{4}$$

LS	RS
$\frac{x-1}{2} + \frac{2x+3}{4}$	3
$\frac{11-1}{4} - 1 + 2\left(\frac{11}{4}\right) + 3$	3
$\frac{11-4}{4} - \frac{4}{4} + \frac{2\left(\frac{11}{2}\right) + 3}{2}$	3
$\frac{7}{4} - \frac{11}{4} + \frac{6}{2} + \frac{3}{2}$	3
$\frac{7}{4} \times \frac{1}{2} + \frac{17}{2} \times \frac{1}{4}$	3
$\frac{7}{8} + \frac{17}{8}$	3
$\frac{24}{8}$	3
3	3
LS	RS

Therefore, $x = \frac{11}{4}$ is a root.

$$\bullet \frac{5}{2x-1} = \frac{3}{x+2}$$

Solution:

$$\frac{5}{2x-1} = \frac{3}{x+2}$$

Before you attempt to solve an equation involving rational expressions with variables in the denominator, you will need to define the nonpermissible values for the equation. You know that division by zero is not permissible. The denominator of a rational expression may never have a value of zero. To find the nonpermissible values, let the denominators equal zero and solve for the variable. For this equation, the nonpermissible values are determined by letting $2x - 1 = 0$ and $x + 2 = 0$.

$$\text{When } 2x - 1 = 0,$$

$$2x = 1$$

$$x = \frac{1}{2}$$

$$\text{When } x + 2 = 0,$$

$$x = -2$$

The nonpermissible values are $x = \frac{1}{2}$ and $x = -2$. This means that $x \neq \frac{1}{2}$ or $x \neq -2$.

Now solve the equation.

$$\frac{5}{2x-1} = \frac{3}{x+2}$$

The L.C.D. is $(2x-1)(x+2)$.

Multiply both sides by the L.C.D. and eliminate the denominators.

$$(2x-1)(x+2)\left(\frac{5}{2x-1}\right) = (2x-1)(x+2)\left(\frac{3}{x+2}\right)$$

$$(x+2)(5) = (2x-1)(3)$$

$$5x + 10 = 6x - 3$$

$$10 = x - 3$$

$$13 = x$$

Is this solution a permissible value? The answer is yes.
The solution to the equation is $x = 13$.

As stated before, some equations involving rational expressions can be changed to quadratic equations. Look at the next example to see how to solve these equations.

Example 5

Solve the equation $\frac{2x-9}{x-7} + \frac{x}{2} = \frac{5}{x-7}$.

Solution:

Determine the nonpermissible values.

$$\text{Let } x - 7 = 0$$

$$x = 7$$

The nonpermissible value is 7 ($x \neq 7$).

Now solve the equation.

$$\frac{2x-9}{x-7} + \frac{x}{2} = \frac{5}{x-7}$$

$$\text{The L.C.D. is } 2(x-7).$$

Multiply both sides by the L.C.D. to eliminate the denominators.

$$2(x-7) \left(\frac{2x-9}{x-7} + \frac{x}{2} \right) = 2(x-7) \left(\frac{5}{x-7} \right)$$
$$2(x-7) \left(\frac{2x-9}{x-7} + \frac{1}{2}(x-7) \right) = 2(x-7) \left(\frac{5}{x-7} \right)$$

$$2(2x-9) + (x-7)(x) = 2(5)$$

$$4x - 18 + x^2 - 7x = 10$$

$$x^2 - 3x - 18 = 10$$

Factor this quadratic equation.

$$x^2 - 3x - 28 = 0$$

$$(x-7)(x+4) = 0$$

$$x-7=0 \text{ or } x+4=0$$

$$x=7 \quad x=-4$$

The value 7 is nonpermissible; therefore, it is not considered to be a root.

The solution to the equation is $x = -4$.

The next example involves irrational number roots.

Example 6

Solve the equation $\frac{4x^2}{x+2} - \frac{10}{x+2} + 4 = 0$.

Solution:

$$x+2=0$$

$$x=-2$$

The nonpermissible value is -2 ($x \neq -2$).

$$\frac{4x^2}{x+2} - \frac{10}{x+2} + 4 = 0$$

The L.C.D. is $x+2$.

$$(x+2) \left(\frac{4x^2}{x+2} - \frac{10}{x+2} + 4 \right) = (x+2)(0)$$

$$(x+2) \left(\frac{4x^2}{x+2} - (x+2) \right) + (x+2)(4) = 0 \quad (\text{Eliminate the denominators.})$$

$$4x^2 - 10 + 4x + 8 = 0$$

$$4x^2 + 4x - 2 = 0 \quad (\text{Divide both sides by 2.})$$

$$2x^2 + 2x - 1 = 0$$

Use the quadratic formula to solve the equation.

$$a = 2, b = 2, c = -1$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(2)(-1)}}{2(2)}$$

$$x = \frac{-2 \pm \sqrt{4+8}}{4}$$

$$x = \frac{-2 \pm \sqrt{12}}{4}$$

$$x = \frac{-2 \pm 2\sqrt{3}}{4}$$

$$x = \frac{-1 \pm \sqrt{3}}{2}$$

$$x = \frac{-1 \pm \sqrt{3}}{2}$$

The roots are $\frac{-1+\sqrt{3}}{2}$ and $\frac{-1-\sqrt{3}}{2}$. Both roots are permissible.

The remainder of this topic deals with solving problems that involve radical equations or equations with rational expressions. Pay careful attention to each example.

Example 7

What number must be added to the numerator and the denominator of the fraction $\frac{3}{7}$ to produce a new fraction equal to $\frac{3}{4}$.

Solution:

Let n be the number to be added.

$$\frac{3+n}{7+n} = \frac{3}{4}$$

The L.C.D. is $4(7+n)$.

The nonpermissible value is -7 ($n \neq -7$).

$$4 \left(\frac{3+n}{7+n} \right) \left(\frac{1}{1} \right) = 4 \left(\frac{3}{4} \right) \left(\frac{1}{1} \right)$$

(Multiply both sides by the L.C.D. to eliminate the denominators.)

$$4(3+n) = (7+n)3$$

$$12 + 4n = 21 + 3n$$

$$12 + n = 21$$

$$n = 9$$

The number to be added is 9.

This can be verified as shown.

Check:

$$n = 9$$

LS	RS
$\frac{3+n}{7+n}$	$\frac{3}{4}$
$\frac{3+9}{7+9}$	$\frac{3}{4}$
$\frac{12}{16}$	$\frac{3}{4}$
$\frac{3}{4}$	$\frac{3}{4}$
LS	RS

The solution is 9.

Example 8

A number plus its reciprocal gives a sum of 8. What is the number?

Solution:

Let x be the number.

Thus, $x + \frac{1}{x} = 8$, $x \neq 0$. The L.C.D. is x .

Example 9

The velocity v , in metres per second, of a steel ball falling through a dense medium is given by the relation $v = \sqrt{3t^2 + 2t + 5}$, where t is the time in seconds. Calculate the time needed in order for the ball to attain a velocity of 12 m/s. Express your answer as a decimal rounded to two decimal places.

Solution:

Substitute $v = 12$ in the relation.

$$12 = \sqrt{3t^2 + 2t + 5}$$

$$(12)^2 = (\sqrt{3t^2 + 2t + 5})^2 \quad (\text{Square both sides.})$$

$$144 = 3t^2 + 2t + 5$$

$$0 = 3t^2 + 2t - 139$$

(Put the quadratic equation in standard form.)

The equation is not factorable; therefore, use the quadratic formula.

$$a = 3, b = 2, c = -139$$

$$\sqrt{60} = \sqrt{4 \times 15}$$

$$= 2\sqrt{15}$$

$$x\left(x + \frac{1}{x}\right) = x(8)$$

$$x(x) + x\left(\frac{1}{x}\right) = 8x$$

$$x^2 + 1 = 8x$$

$$x^2 - 8x + 1 = 0$$

Use the quadratic formula to solve.

$$a = 1, b = -8, c = 1$$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{8 \pm \sqrt{64 - 4}}{2}$$

$$x = \frac{8 \pm \sqrt{60}}{2}$$

$$x = \frac{8 \pm 2\sqrt{15}}{2}$$

$$x = \frac{4 \pm \sqrt{15}}{1}$$

$$x = 4 \pm \sqrt{15}$$

The numbers are $4 + \sqrt{15}$ and $4 - \sqrt{15}$.

$$t = \frac{-2 \pm \sqrt{(-2)^2 - 4(3)(-139)}}{2(3)}$$

$$t = \frac{-2 \pm \sqrt{4 + 1668}}{6}$$

$$t = \frac{-2 \pm \sqrt{1672}}{6}$$

$$t \doteq \frac{-2 \pm 40.89}{6}$$

$$t \doteq \frac{-2 + 40.89}{6} \quad \text{or} \quad t \doteq \frac{-2 - 40.89}{6}$$

$$t \doteq 6.48 \quad t \doteq -7.15$$

The negative root is extraneous.

The ball would take 6.48 s to reach a velocity of 12 m/s.

Do the questions that follow.

1. What is a radical equation?
2. Why must all possible solutions to a radical equation be verified?

3. Solve any five of the following radical equations. Remember to verify all possible solutions.

a. $x + \sqrt{x + 3} = 3$

b. $\sqrt{2x + 4} + 3 = x + 5$

c. $4 - \sqrt{3x - 14} = 7 - x$

d. $x - \sqrt{x} = 6$

e. $\sqrt{3z + 6} - z = -4$

f. $x - 2\sqrt{x - 5} = 5$

g. $a + \sqrt{a + 9} = 2a - 3$

h. $x + 3 = \sqrt{2x + 19}$

4. Before you attempt to solve an equation with rational expressions, what should you first identify?
5. How do you obtain the nonpermissible values for an equation with rational expressions?

Note that -7.15 is an inadmissible value since time cannot be a negative quantity.

6. Solve any five of the following equations. Be sure to identify all nonpermissible values.

a. $\frac{4}{x} + \frac{x}{4} = \frac{5}{2}$

b. $\frac{3}{x+2} = \frac{1}{4} - \frac{1}{x-4}$

c. $\frac{5x+7}{3x+2} = \frac{x-1}{x-2}$

d. $\frac{1-5x}{-3x} = \frac{x+1}{2}$

e. $\frac{6}{x+1} - \frac{5}{1-x} = 7$

f. $\frac{t+2}{t} - \frac{4t}{3} + 2 = 0$

g. $\frac{n+1}{2} - \frac{n-1}{3} = \frac{2n+3}{4n}$

h. $\frac{1-x}{2x} = \frac{5x}{3x+1}$

i. $x - \frac{2x-1}{3} = x^2 - \frac{1}{2}$

j. $\frac{x+3}{2x-7} = \frac{2x-1}{x-3}$

7. Do any three of the following problems.

- a. If a certain number is decreased by 28 times its reciprocal, the result is 3. What is the number?

- b. In coordinate geometry the distance d between two points can be found by using the formula $d = \sqrt{(x-h)^2 + (y-k)^2}$. If h has a value of 3, y has a value of 2, k has a value of -2 , and d is 5 units, find x .

- c. The equation for a function is given by $y = \sqrt{2x^2 - 5x + 33}$. Find x when y has a value of 6.

- d. A student wrote three math exams obtaining marks of 72%, 56%, and 86%. What mark is needed on the next exam to obtain an average of 75% on all four exams?

e. Solve the equation $\sqrt{\sqrt{x^2 - 3x}} = \sqrt{2x}$.



For solutions to Activity 1, turn to Appendix A, Topic 6.



If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

You may decide to do both.



Extra Help

Some people learn best by doing. The old saying "Practice makes perfect" definitely has many applications in mathematics. The following exercises are designed to give you a better understanding of radical equations and equations with rational expressions by giving you more practice. Try the following.

1. Solve and verify the following radical equations. Do at least five.

a. $\sqrt{\frac{x}{6}} + 4 = 8$

b. $\sqrt{3x} = \frac{2}{3}$

c. $\sqrt{5x+3} + 7 = 9$

d. $\sqrt{7x-9} = \sqrt{2x+21}$

e. $\sqrt{x^2+3x-2} = 0$

f. $\sqrt{3x+10} - x = 0$

g. $\sqrt{x-3} + 3 = x$

h. $x + 2 = \sqrt{18-x}$

i. $\sqrt{3x-5} - x = -5$

j. $\sqrt{7x+25} = 1+x$

k. $\sqrt{2x-7} = 4$

l. $3\sqrt{x} = x-4$

2. Solve the following equations. Do at least five.

a. $\frac{x}{x+4} = \frac{3}{x-1}$

b. $\frac{6}{x+2} + \frac{1}{x-2} = 1$

c. $\frac{3}{x} + \frac{2}{x-1} = 2$

d. $2 + \frac{3}{x-5} = \frac{x}{x+3}$

e. $\frac{x-1}{x+1} - \frac{6}{x-3} = 3$

f. $\frac{2}{x+3} + \frac{3}{x+4} = \frac{7}{(x+3)(x+4)}$

g. $x + \frac{4}{x} = 8$

h. $\frac{3x-2}{x+5} = \frac{x-5}{3x+2}$

i. $\frac{4}{x-5} + \frac{1}{x+2} = \frac{2x+7}{(x-5)(x+2)}$



For solutions to Extra Help, turn to Appendix A, Topic 6.



Extensions

The radical equations that you solved in this topic were solved by first isolating the radical and then squaring both sides to eliminate the radical sign. Many radical equations must be solved by squaring both sides of the equation twice. The next example shows how this is done.

Example 10

Solve the equation $\sqrt{2x+3} - \sqrt{x+1} = 1$.

Solution:

Isolate one of the radicals.

$$\sqrt{2x+3} - \sqrt{x+1} = 1$$

$$\sqrt{2x+3} = 1 + \sqrt{x+1}$$

(Try to isolate the larger radical.)

$$(\sqrt{2x+3})^2 = (1 + \sqrt{x+1})^2$$

(Square both sides.)

$$2x+3 = 1 + 2\sqrt{x+1} + x+1$$

Isolate the remaining radical.

$$2x+3 = 2\sqrt{x+1} + x+2$$

$$x+1 = 2\sqrt{x+1}$$

$$(x+1)^2 = (2\sqrt{x+1})^2$$

(Square both sides.)

$$\begin{aligned} & (1 + \sqrt{x+1})^2 \\ &= (1 + \sqrt{x+1})(1 + \sqrt{x+1}) \\ &= 1 + \sqrt{x+1} + \sqrt{x+1} + x+1 \\ &= 1 + 2\sqrt{x+1} + x+1 \end{aligned}$$

$$x^2 + 2x + 1 = 4(x + 1)$$

$$x^2 + 2x + 1 = 4x + 4$$

$$x^2 - 2x - 3 = 0$$

(This is a quadratic equation.)

$$(x - 3)(x + 1) = 0$$

(Factor.)

$$x - 3 = 0 \text{ or } x + 1 = 0$$

$$x = 3 \quad x = -1$$

Check:

$$x = 3$$

$$x = -1$$

LS	RS	LS	RS
$\sqrt{2x+3} - \sqrt{x+1}$	1	$\sqrt{2x+3} - \sqrt{x+1}$	1
$\sqrt{2(3)+3} - \sqrt{3+1}$	1	$\sqrt{2(-1)+3} - \sqrt{-1+1}$	1
$\sqrt{6+3} - \sqrt{4}$	1	$\sqrt{-2+3} - \sqrt{0}$	1
$\sqrt{9} - 2$	1	$\sqrt{1} - 0$	1
3 - 2	1	1 - 0	1
1	1	1	1
LS =	RS	LS =	RS

The roots of the equation are 3 and -1.

Do the following exercises.

Find the solution to each equation.

$$1. \sqrt{x^2 - 22} = \sqrt{3\sqrt{x}}$$

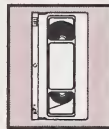
$$2. \sqrt{3x} - \sqrt{x-3} = 3$$

$$3. \sqrt{x+4} + \sqrt{x-4} = 4$$

$$4. \sqrt{2x-3} - \sqrt{x-2} = 1$$



For solutions to **Extensions**, turn to **Appendix A, Topic 6**.



If you have access to a VCR (videocassette recorder) you may wish to view the program titled **Graphing Quadratic Functions** as a review of some of the major topics of this unit. This is program 6 on the videocassette titled *Graphing Mathematical Concepts*.¹

¹ *Graphing Mathematical Concepts* is a title of ACCESS Network.

Unit Summary



What You Have Learned

In this unit you studied how to

- sketch the graphs of quadratic functions in the following forms:

$$y = ax^2$$

$$y = ax^2 + k$$

$$y = a(x - h)^2$$

$$y = a(x - h)^2 + k$$

- identify the coordinates of the vertex of a parabola
- change standard form into completed square form
- write the equation for a quadratic function by inspecting its graph
- identify maximum and minimum values

Unit Summary

- write the equation for the axis of symmetry
- determine the domain and the range for a quadratic function
- calculate the y -intercept for a quadratic function
- apply the concepts you learned in solving maximum or minimum problems
- calculate the x -intercept(s) for a quadratic function
- solve quadratic equations
- use the quadratic formula to solve quadratic equations
- solve radical equations and equations with rational expressions both of which can be simplified to quadratic equations
- solve problems that can be represented by quadratic equations

You are now ready to
complete the **Unit Assignment**.

Appendices



Appendix A Solutions

Review

Topic 1 Sketching Quadratic Functions

Topic 2 Completing the Square

Topic 3 Characteristics of Quadratic Functions

Topic 4 Solving Quadratic Equations

Topic 5 The Quadratic Formula

Topic 6 Extensions of Quadratic Equations



Appendix B Graphing Material

Graph Paper

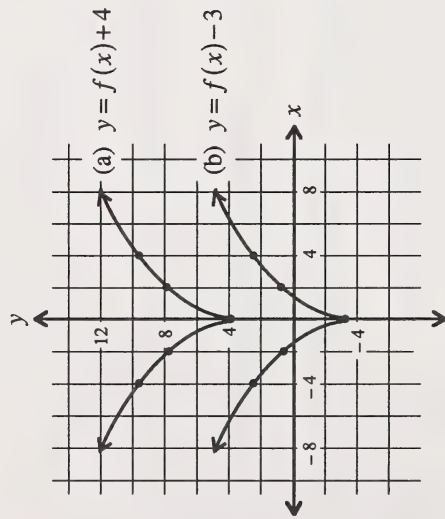


Appendix A Solutions

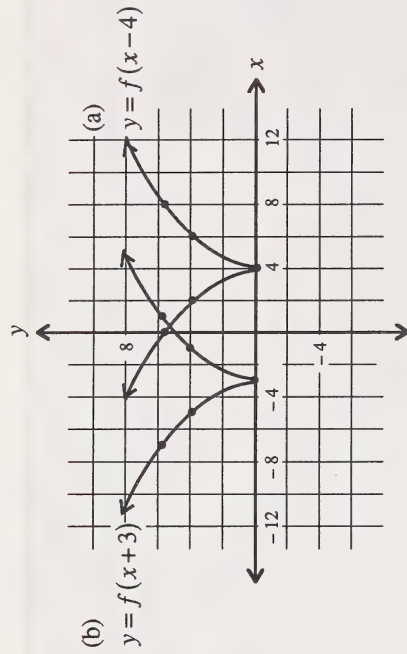


Review

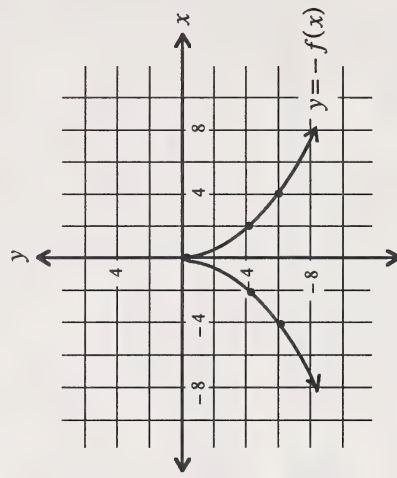
1.



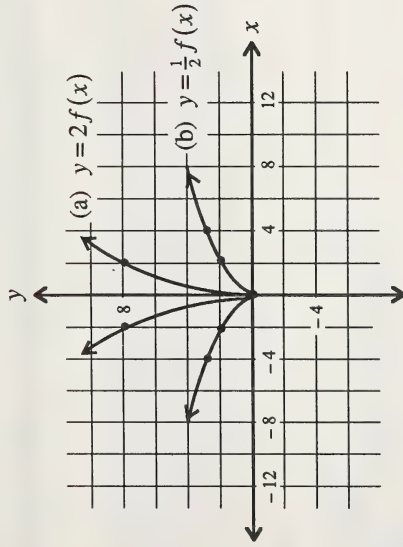
2.



3.



4.



5. a. The equation of the horizontal line is $y = 3$.
The equation of the vertical line is $x = 5$.
- b. The equation of the horizontal line is $y = 5$.
The equation of the vertical line is $x = -2$.

6. a. $x^2 - 6x + 9 = (x - 3)^2$

b. $x^2 + 2x + 1 = (x + 1)^2$

c. $x^2 + \frac{4}{3}x + \frac{4}{9} = \left(x + \frac{2}{3}\right)^2$

d. $x^2 - \frac{4}{3}x + \frac{4}{9} = \left(x - \frac{2}{3}\right)^2$

7. a. $5x^2 - 10y = 5(x^2 - 2y)$

b. $2x^2 - 10x = 2x(x - 5)$

c. $x^2 - 25 = (x + 5)(x - 5)$

d. $4x^2 - 9 = (2x + 3)(2x - 3)$

e. $x^2 + 2x - 15 = (x + 5)(x - 3)$

f. $x^2 - 5x - 14 = (x - 7)(x + 2)$

g. $3x^2 - 8x + 4 = (3x - 2)(x - 2)$

h. $2x^2 - x - 6 = (2x + 3)(x - 2)$

8. a. $\sqrt{18} = \sqrt{9 \times \sqrt{2}}$
 $= 3\sqrt{2}$

b. $2\sqrt{32} = 2\sqrt{16 \times \sqrt{2}}$
 $= 2 \times 4 \times \sqrt{2}$
 $= 8\sqrt{2}$

c. $\sqrt{8 + \sqrt{12}} = \sqrt{4 \times \sqrt{2}} + \sqrt{4 \times \sqrt{3}}$
 $= 2\sqrt{2} + 2\sqrt{3}$

d. $3\sqrt{27} = 3\sqrt{9 \times \sqrt{3}}$
 $= 3 \times 3 \times \sqrt{3}$
 $= 9\sqrt{3}$

9. a. $2x + 5 = 0$

$$2x = -5$$

$$x = -\frac{5}{2}$$

Therefore, $x = -\frac{5}{2}$ is a nonpermissible value.

b. $x - 4 = 0$

$$x = 4$$

Therefore, $x = 4$ is a nonpermissible value.

c. $2x = 0$

$$x = 0$$

Therefore, $x = 0$ is a nonpermissible value.



Exploring Topic 1

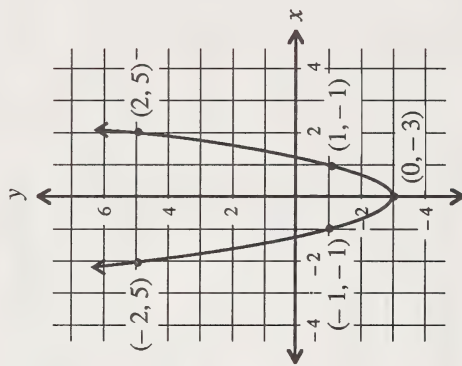
Activity 1

Sketch the graphs of quadratic functions written in the form

$$y = a(x - h)^2 + k.$$

1. $y = 2x^2 - 3$

x	-2	-1	0	1	2
y	5	-1	-3	-1	5



2. A quadratic function is of degree two.

3. A parabola is the characteristic shape.

4. The equation of the reference function is $y = x^2$.

5. The vertex is the point at which the parabola changes direction.

6. The axis of symmetry divides a parabola into two congruent parts, each of which is a mirror reflection of the other.

7. The coordinates of the vertex are $(0, 0)$. The axis of symmetry is the y -axis described by the equation $x = 0$.

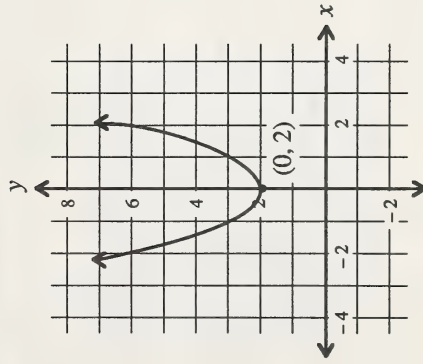
8. a. The graph of $y = x^2 + 8$ is the same as the graph of $y = x^2$ translated eight units upward.

- b. The graph of $y = x^2 - 5$ is the same as the graph of $y = x^2$ translated five units downward.

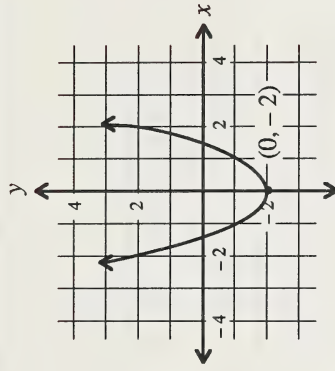
9. a. $(0, 1)$

- b. $(0, -4)$

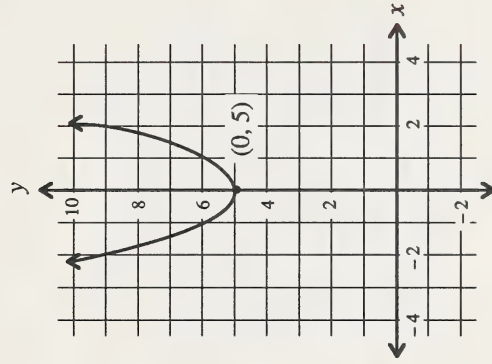
10. a. $y = x^2 + 2$



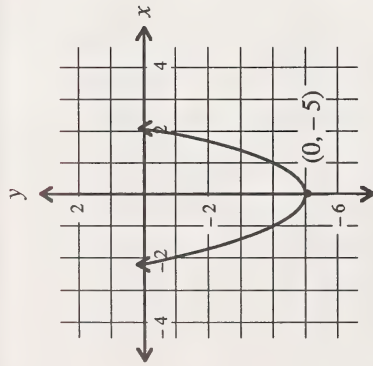
- b. $x^2 - 2$



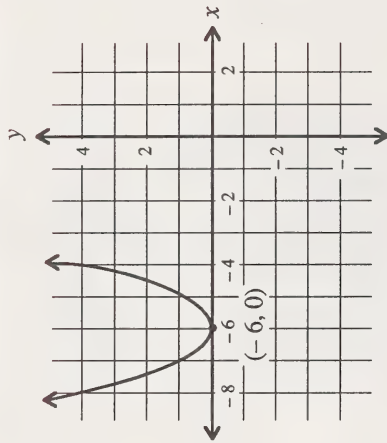
- c. $y = x^2 + 5$



d. $y = x^2 - 5$



16. a. $y = (x + 6)^2$



11. The graph of $y = x^2 + k$ is a vertical translation of the graph of $y = x^2$ by k units.

12. The graph of $y = (x - h)^2$ is a horizontal translation of the graph of $y = x^2$ by h units.

13. a. The graph of $y = (x - 7)^2$ is the same as the graph of $y = x^2$ translated seven units to the right.

- b. The graph of $y = (x + 10)^2$ is the same as the graph of $y = x^2$ translated ten units to the left.

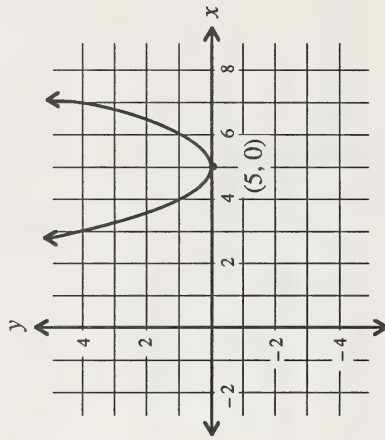
14. a. $(-3, 0)$

b. $(2, 0)$

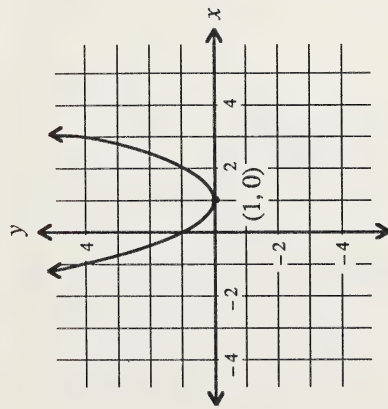
15. a. $(h, 0)$

b. $(-h, 0)$

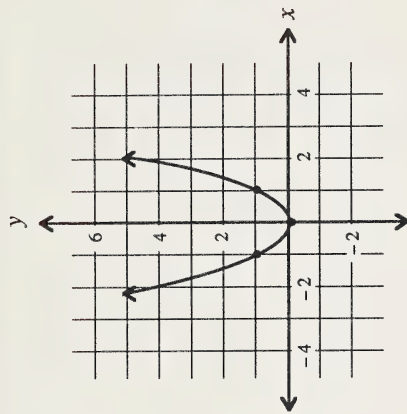
b. $y = (x - 5)^2$



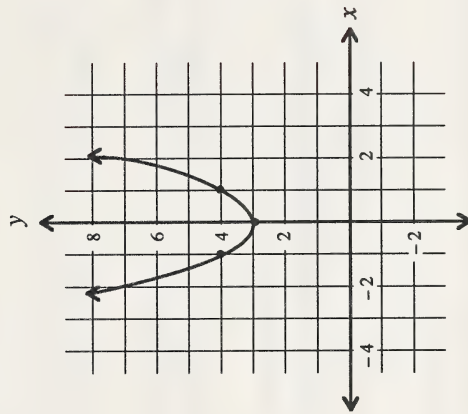
c. $y = (x-1)^2$



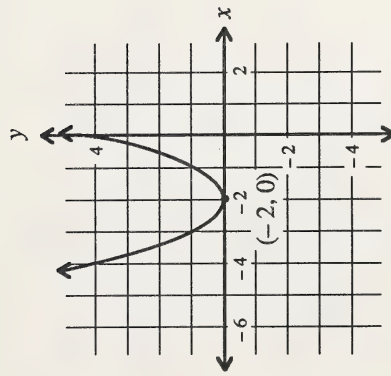
17. a. $y = x^2$



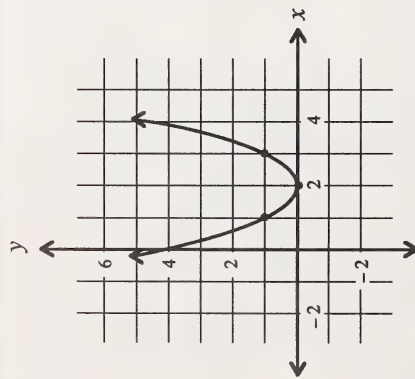
b. $y = x^2 + 3$



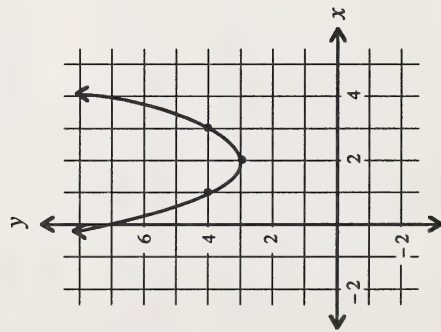
d. $y = (x+2)^2$



c. $y = (x - 2)^2 + 3$



d. $y = (x - 2)^2 + 3$



18. The graph of $y = (x - 2)^2 + 3$ is congruent to the graph of $y = x^2$ translated three units upward and two units to the right. The vertex is at $(2, 3)$.

19. The graph of $y = (x - h)^2 + k$ is congruent to the graph of $y = x^2$ translated k units upward and h units to the right. The vertex should be at (h, k) .

20. a. The graph of $y = (x - 3)^2 + 2$ is the graph of $y = x^2$ translated three units to the right and two units up.

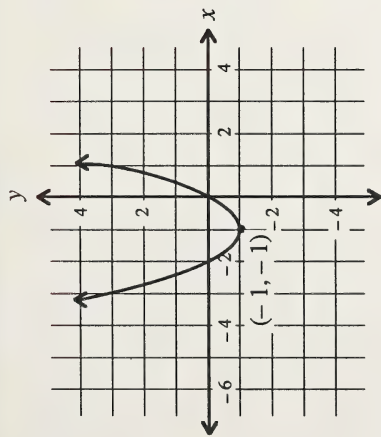
b. The graph of $y = (x + 8)^2 - 5$ is the graph of $y = x^2$ translated eight units to the left and five units down.

21. a. $(8, -3)$

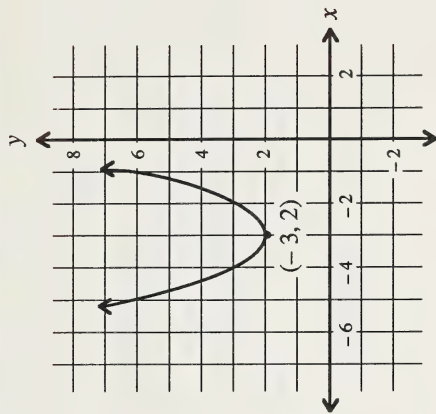
b. $(-7, 14)$

c. (h, k)

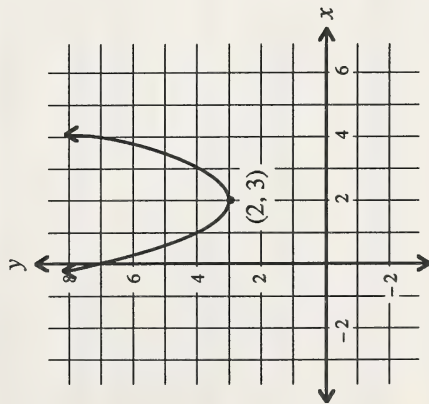
22. a. $y = (x+1)^2 - 1$



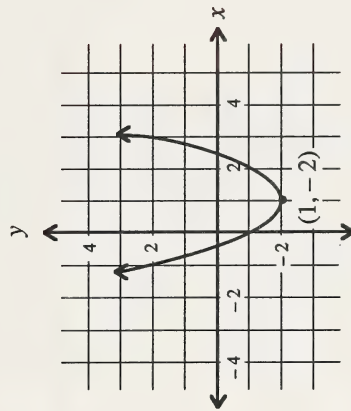
c. $y = (x+3)^2 + 2$



b. $y = (x-2)^2 + 3$



d. $y = (x-1)^2 - 2$



23. The graph of $y = 3x^2$ opens upward. The graph of $y = -3x^2$ opens downward.

24. The value of a determines the direction of opening and the width of the parabola.

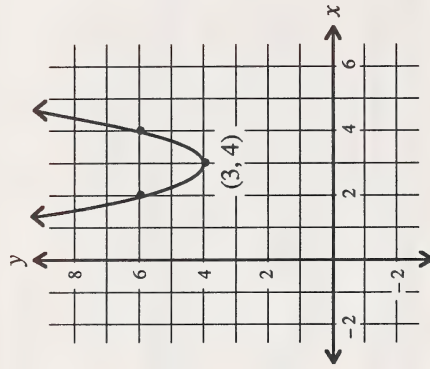
25. a. The graph of $y = 8x^2$ is narrower than the graph of $y = x^2$.

b. The graph of $y = \frac{1}{5}x^2$ is wider than the graph of $y = x^2$.

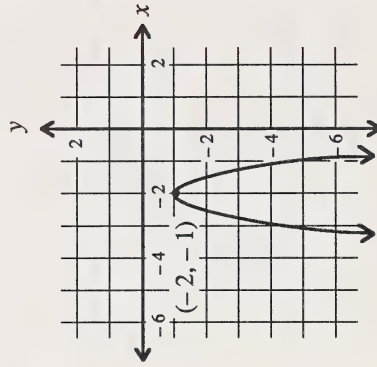
c. The graph of $y = -3x^2$ is a reflection of $y = x^2$ in the x -axis and it is narrower than the graph of $y = x^2$.

d. The graph of $y = -\frac{1}{4}x^2$ is a reflection of $y = x^2$ in the x -axis and it is wider than the graph of $y = x^2$.

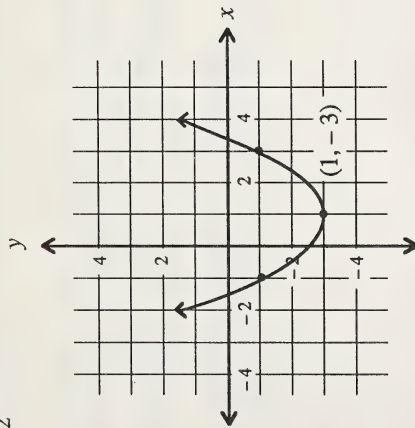
26. a. $y = 2(x-3)^2 + 4$



b. $y = -3(x+2)^2 - 1$



c. $y = \frac{1}{2}(x-1)^2 - 3$



27. a. The vertex is at $(3, 3)$.

$a = -3$

$y = -3(x-3)^2 + 3$

b. The vertex is at $(1, -1)$.

$a = -\frac{1}{2}$

$y = -\frac{1}{2}(x-1)^2 - 1$

c. The vertex is at $(-2, 4)$.

$a = -1$

$y = -1(x+2)^2 + 4$

Extra Help

1. a. $(0, 0)$

b. $(0, 0)$

c. $(0, 2)$

d. $(0, -2)$

e. $(5, 0)$

f. $(-2, 0)$

g. $(1, 0)$

h. $(-2, -3)$

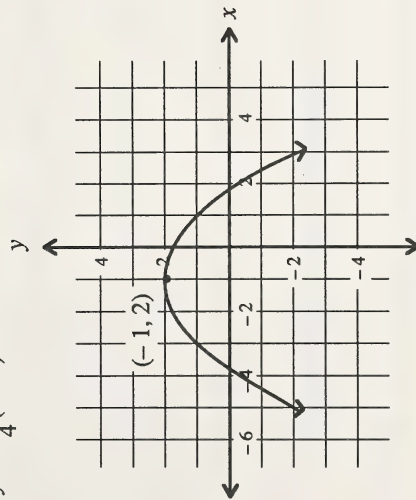
i. $(7, 19)$

j. $(-19, -72)$

k. $(475, -189)$

l. $(-59, 0)$

d. $y = -\frac{1}{4}(x+1)^2 + 2$



2. a. The graph opens downward and is narrower than the reference parabola.
- b. The graph opens upward and is wider than the reference parabola.
- c. The graph opens upward and is narrower than the reference parabola.
- d. The graph opens downward and is wider than the reference parabola.
- e. The graph opens downward and is the same width as the reference parabola.
- f. The graph opens downward and is narrower than the reference parabola.
- g. The graph opens upward and is narrower than the reference parabola. Note that $\frac{3}{2} = 1\frac{1}{2}$ which is greater than 1.

Extensions

1. Each point of the new parabola would be located 565 units above the corresponding point of the $y = x^2$ parabola.
2. Each point of the new parabola would be located 10 000 or 10^4 units to the right of the corresponding point of the $y = x^2$ parabola.

3. Each point of the new parabola would be located 64 or 4^3 units to the left and 3125 or 5^5 units below the corresponding point of the $y = x^2$ parabola.

4. a. $(0, 565)$ b. $(10\,000, 0)$

- c. $(-64, -3125)$

5. Since $a = 1000$, the parabola would open upward, be very narrow, and have its vertex at $(2, 6)$.

6. Since $a = -5^{-4} = -\frac{1}{5^4} = -\frac{1}{625}$, the parabola would open downward, be very wide, and have its vertex at $(-2, -3)$.



Exploring Topic 2

Activity 1

Change quadratic functions in the form $y = ax^2 + bx + c$ to the form $y = a(x - h)^2 + k$ by completing the square.

1. A quadratic function is any function that can be written in the form $y = ax^2 + bx + c$ where $a, b, c \in R$ and $a \neq 0$.

2. The coordinates of the vertex are easily identifiable in the form

$$y = a(x-h)^2 + k.$$

3. The word quadratic implies second degree. All quadratic functions are of second degree.

4. The standard form for a quadratic function is $y = ax^2 + bx + c$.

5. The completed square form for a quadratic function is

$$y = a(x-h)^2 + k.$$

6. a. $y = 3x^2 + 2x$

$$a = 3, b = 2, c = 0$$

b. $y = -3x^2 - 5$

$$a = -3, b = 0, c = -5$$

c. $y = -4x^2 + 5x - 3$

$$a = -4, b = 5, c = -3$$

d. $y = 3x^2 + 2x - 5$

$$a = 3, b = 2, c = -5$$

7. A perfect square trinomial is a trinomial that can be factored into a binomial that is squared.

8. The relationship is $\left(\frac{1}{2} \times M\right)^2 = N$.

9. a. $\left(\frac{1}{2} \times 12\right)^2 = (6)^2$
 $= 36$

b. $\left(\frac{1}{2} \times 18\right)^2 = 9^2$
 $= 81$

c. $\left(\frac{1}{2} \times 5\right)^2 = \left(\frac{5}{2}\right)^2$
 $= \frac{25}{4}$

d. $\left(\frac{1}{2} \times -8\right)^2 = (-4)^2$
 $= 16$

e. $\left(\frac{1}{2} \times -3\right)^2 = \left(\frac{-3}{2}\right)^2$
 $= \frac{9}{4}$

f. $\left(\frac{1}{2} \times -1\right)^2 = \left(\frac{-1}{2}\right)^2$
 $= \frac{1}{4}$

10. a. $(x-7)^2$

b. $(x-1)^2$

c. $(x+2)^2$

d. $\left(x + \frac{7}{3}\right)^2$

e. $\left(x - \frac{1}{2}\right)^2$

f. $\left(x + \frac{3}{4}\right)^2$

11. a. $y = x^2 - 8x$

$$y = (x^2 - 8x + 16) - 16$$

$$y = (x - 4)^2 - 16$$

The vertex is at $(4, -16)$.

b. $y = x^2 + 18x$

$$y = (x^2 + 18x + 81) - 81$$

$$y = (x + 9)^2 - 81$$

The vertex is at $(-9, -81)$.

c. $y = x^2 - 3x$

$$y = \left(x^2 - 3x + \frac{9}{4}\right) - \frac{9}{4}$$

$$y = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}$$

The vertex is at $\left(\frac{3}{2}, -\frac{9}{4}\right)$.

12. a. $y = x^2 - 10x + 10$

$$y = (x^2 - 10x + 25) + 10 - 25$$

$$y = (x - 5)^2 - 15$$

The vertex is at $(5, -15)$.

b. $y = x^2 - 3x + 7$

$$y = \left(x^2 - 3x + \frac{9}{4}\right) + 7 - \frac{9}{4}$$

$$y = \left(x - \frac{3}{2}\right)^2 + \frac{28}{4} - \frac{9}{4}$$

$$y = \left(x - \frac{3}{2}\right)^2 + \frac{19}{4}$$

The vertex is at $\left(\frac{3}{2}, \frac{19}{4}\right)$.

c. $y = x^2 + 2x + 5$

$$y = (x^2 + 2x + 1) + 5 - 1$$

$$y = (x + 1)^2 + 4$$

The vertex is at $(-1, 4)$.

13. a. $y = 2x^2 - 5x - 6$

$$y = 2\left(x^2 - \frac{5}{2}x\right) - 6$$

$$y = 2\left(x^2 - \frac{5}{2}x + \frac{25}{16}\right) - 6 - 2\left(\frac{25}{16}\right)$$

$$y = 2\left(x - \frac{5}{4}\right)^2 - 6 - \frac{25}{8}$$

$$y = 2\left(x - \frac{5}{4}\right)^2 - \frac{73}{8}$$

The vertex is at $\left(\frac{5}{4}, -\frac{73}{8}\right)$.

b. $y = 3x^2 - 6x + 10$

$$y = 3\left(x^2 - 2x\right) + 10$$

$$y = 3\left(x^2 - 2x + 1\right) + 10 - 3$$

$$y = 3(x-1)^2 + 7$$

The vertex is at $(1, 7)$.

c. $y = 5x^2 - 15x + 10$

$$y = 5\left(x^2 - 3x\right) + 10$$

$$y = 5\left(x^2 - 3x + \frac{9}{4}\right) + 10 - 5\left(\frac{9}{4}\right)$$

$$y = \left(x - \frac{3}{2}\right)^2 + 10 - \frac{45}{4}$$

$$y = \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$$

The vertex is at $\left(\frac{3}{2}, -\frac{5}{4}\right)$.

14. a. $y = -2x^2 + 16x - 29$

$$y = -2\left(x^2 - 8x\right) - 29$$

$$y = -2\left(x^2 - 8x + 16\right) - 29 + 32$$

$$y = -2(x-4)^2 + 3$$

The vertex is at $(4, 3)$.

b. $y = -3x^2 - 6x + 4$

$$y = -3\left(x^2 + 2x\right) + 4$$

$$y = -3\left(x^2 + 2x + 1\right) + 4 + 3$$

$$y = -3(x+1)^2 + 7$$

The vertex is at $(-1, 7)$.

$$c. \quad y = -x^2 + 2x - 1$$

$$y = -1(x^2 - 2x) - 1$$

$$y = -1(x^2 - 2x + 1) - 1 + 1$$

$$y = -1(x-1)^2 + 0$$

The vertex is at $(1, 0)$.

Extra Help

$$1. \quad y = x^2 + 9x$$

$$y = \left(x^2 + 9x + \frac{81}{4}\right) - \frac{81}{4}$$

$$y = \left(x + \frac{9}{2}\right)^2 - \frac{81}{4}$$

The vertex is at $\left(-\frac{9}{2}, -\frac{81}{4}\right)$.

$$2. \quad y = x^2 - 12x$$

$$y = (x^2 - 12x + 36) - 36$$

$$y = (x-6)^2 - 36$$

The vertex is at $(6, -36)$.

$$3. \quad y = x^2 + 6x - 3$$

$$y = (x^2 + 6x + 9) - 3 - 9$$

$$y = (x+3)^2 - 12$$

The vertex is at $(-3, -12)$.

$$4. \quad y = x^2 - 8x + 7$$

$$y = (x^2 - 8x + 16) + 7 - 16$$

$$y = (x-4)^2 - 9$$

The vertex is at $(4, -9)$.

$$5. \quad y = 3x^2 - 15x + 1$$

$$y = (3x^2 - 15x) + 1$$

$$y = 3(x^2 - 5x) + 1$$

$$y = 3\left(x^2 - 5x + \frac{25}{4}\right) + 1 - 3\left(\frac{25}{4}\right)$$

$$y = 3\left(x - \frac{5}{2}\right)^2 + 1 - \frac{75}{4}$$

$$y = 3\left(x - \frac{5}{2}\right)^2 - \frac{71}{4}$$

The vertex is at $\left(\frac{5}{2}, -\frac{71}{4}\right)$.

$$6. y = 2x^2 + 5x - 3$$

$$y = \left(2x^2 + 5x\right) - 3$$

$$y = 2\left(x^2 + \frac{5}{2}x\right) - 3$$

$$y = 2\left(x^2 + \frac{5}{2}x + \frac{25}{16}\right) - 3 - 2\left(\frac{25}{16}\right)$$

$$y = 2\left(x + \frac{5}{4}\right)^2 - 3 - \frac{25}{8}$$

$$y = 2\left(x + \frac{5}{4}\right)^2 - \frac{49}{8}$$

The vertex is at $\left(-\frac{5}{4}, -\frac{49}{8}\right)$.

$$7. y = -2x^2 - 8x + 1$$

$$y = \left(-2x^2 - 8x\right) + 1$$

$$y = -2\left(x^2 + 4x\right) + 1$$

$$y = -2\left(x^2 + 4x + 4\right) + 1 - (-2)(4)$$

$$y = -2(x+2)^2 + 1 + 8$$

$$y = -2(x+2)^2 + 9$$

The vertex is at $(-2, 9)$.

$$8. y = -3x^2 + x - 2$$

$$y = \left(-3x^2 + x\right) - 2$$

$$y = -3\left(x^2 - \frac{x}{3}\right) - 2$$

$$y = -3\left(x^2 - \frac{1}{3}x + \frac{1}{36}\right) - 2 - (-3)\left(\frac{1}{36}\right)$$

$$y = -3\left(x - \frac{1}{6}\right)^2 - 2 + \frac{1}{12}$$

$$y = -3\left(x - \frac{1}{6}\right)^2 - \frac{23}{12}$$

The vertex is at $\left(\frac{1}{6}, -\frac{23}{12}\right)$.

Extensions

$$1. y = 7x^2 - 2x + 1$$

$$y = 7\left(x^2 - \frac{2}{7}x\right) + 1$$

$$y = 7\left(x^2 - \frac{2}{7}x + \frac{1}{49}\right) + 1 - 7\left(\frac{1}{49}\right)$$

$$y = 7\left(x - \frac{1}{7}\right)^2 + 1 - \frac{1}{7}$$

$$y = 7\left(x - \frac{1}{7}\right)^2 + \frac{6}{7}$$

The vertex is at $\left(\frac{1}{7}, \frac{6}{7}\right)$.

$$2. y = \frac{1}{4}x^2 - \frac{2}{3}x + \frac{1}{5}$$

$$y = \left(\frac{1}{4}x^2 - \frac{2}{3}x \right) + \frac{1}{5}$$

$$y = \frac{1}{4} \left(x^2 - \frac{8}{3}x \right) + \frac{1}{5}$$

$$y = \frac{1}{4} \left(x^2 - \frac{8}{3}x + \frac{16}{9} \right) + \frac{1}{5} - \frac{1}{4} \left(\frac{16}{9} \right)$$

$$y = \frac{1}{4} \left(x - \frac{4}{3} \right)^2 + \frac{1}{5} - \frac{4}{9}$$

$$y = \frac{1}{4} \left(x^2 - \frac{4}{3} \right)^2 - \frac{11}{45}$$

The vertex is at $\left(\frac{4}{3}, -\frac{11}{45} \right)$.

$$3. y = 1.72x^2 + 5.16x - 3.82$$

$$y = \left(1.72x^2 + 5.16x \right) - 3.82$$

$$y = 1.72 \left(x^2 + 3x \right) - 3.82$$

$$y = 1.72 \left(x^2 + 3x + 2.25 \right) - 3.82 - 1.72 \times 2.25$$

$$y = 1.72(x + 1.5)^2 - 3.82 - 3.87$$

$$y = 1.72(x + 1.5)^2 - 7.69$$

The vertex is at $(-1.5, -7.69)$.

$$4. y = ax^2 + bx + c$$

$$y = (ax^2 + bx) + c$$

$$y = a \left(x^2 + \frac{b}{a}x \right) + c$$

$$y = a \left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} \right) + c - a \times \frac{b^2}{4a^2}$$

$$y = a \left(x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a}$$

$$y = a \left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a}$$

$$y = a \left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a}$$

The vertex is at $\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a} \right)$.

Can you predict how this result may be used to find the coordinates of the vertex if the function is given in the form

$y = ax^2 + bx + c$? Check your prediction on several problems and see if it works.



Exploring Topic 3

Activity 1

Identify and calculate the characteristics that define quadratic functions.

1. If $a > 0$, the parabola opens upward and a minimum value occurs.
If $a < 0$, the parabola opens downward and a maximum value occurs.
 - a. The value of a is 3 and a minimum occurs.
 - b. The value of a is -3 and a maximum occurs.
 - c. The value of a is -5 and a maximum occurs.
 - d. $y = 5(x-2)^2 + 3$
The value of a is 5 and a minimum occurs.
3. The minimum value is the smallest possible value for the dependent variable.
The maximum value is the largest possible value for the dependent variable.
4. The maximum or minimum value of the function is 6. This maximum or minimum value occurs when the independent variable has a value of 5.

5. When $x = -3$, a minimum value of 8 occurs.
6. a. The function has a maximum value of -3 since $y = -3$.
b. The function has a minimum value of 4 since $y = 4$.
c. The function has a minimum value of -7 since $y = -7$.

7. a. $y = 2x^2 - 8x + 7$
 $y = 2(x^2 - 4x) + 7$
 $y = 2(x^2 - 4x + 4) + 7 - 2(4)$
 $y = 2(x-2)^2 + 7 - 8$
 $y = 2(x-2)^2 - 1$
The function has a minimum value of -1 .

- b. $y = -3x^2 - 6x + 2$
 $y = -3(x^2 + 2x) + 2$
 $y = -3(x^2 + 2x + 1) + 2 - (-3)(1)$
 $y = -3(x+1)^2 + 2 + 3$
 $y = -3(x+1)^2 + 5$
The function has a maximum value of 5.

c. $y = -x^2 + 7x$

$y = -1\left(x^2 - 7x\right)$

$y = -1\left(x^2 - 7x + \frac{49}{4}\right) - (-1)\left(\frac{49}{4}\right)$

$y = -1\left(x - \frac{7}{2}\right)^2 + \frac{49}{4}$

The function has a maximum value of $\frac{49}{4}$.

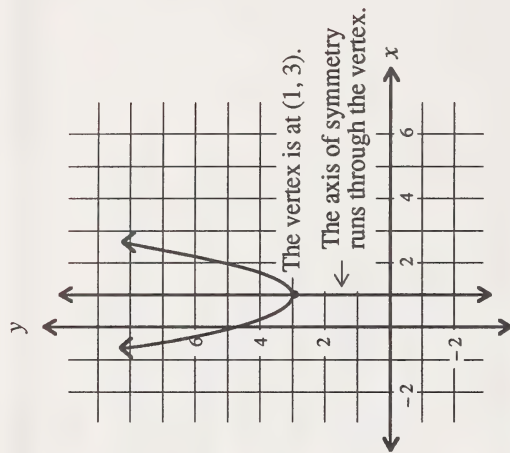
8. The axis of symmetry is a vertical line dividing the parabola into two parts that are mirror images of each other. It can be found running through the vertex.

9. a. $x = 3$

b. $x = -5$

c. $y = 2(x-0)^2 + 6$
 $x = 0$

10.



11. a. Domain: $x \in R$
Range: $y \leq -2$

b. Domain: $x \in R$
Range: $y \geq 6$

c. $y = 5(x-0)^2 - 7$
Domain: $x \in R$
Range: $y \geq -7$

12. From the coordinates of the vertex, you can identify the following:

- maximum or minimum value
- axis of symmetry

13. The y -intercept of a function is the point where the graph of the function crosses the y -axis.

14. a. $y = 5x^2 + 6x$

Let $x = 0$.

$$y = 5(0)^2 + 6(0)$$

$$y = 0 + 0$$

$$y = 0$$

The y -intercept is 0.

b. $y = -2(x+3)^2 - 5$

Let $x = 0$.

$$y = -2(0+3)^2 - 5$$

$$y = -2(3)^2 - 5$$

$$y = -2(9) - 5$$

$$y = -18 - 5$$

$$y = -23$$

The y -intercept is -23 .

c. $y = 2x^2 + 6x - 5$

Let $x = 0$.

$$y = 2(0)^2 + 6(0) - 5$$

$$y = 0 + 0 - 5$$

$$y = -5$$

The y -intercept is -5 .

15. a. $y = -2(x+3)^2 - 1$

i. $(-3, -1)$

ii. The function has a maximum value of -1 .

iii. $x = -3$

iv. $x \in R$

v. $y \leq -1$

vi. Let $x = 0$.

$$y = -2(0+3)^2 - 1$$

$$y = -2(3)^2 - 1$$

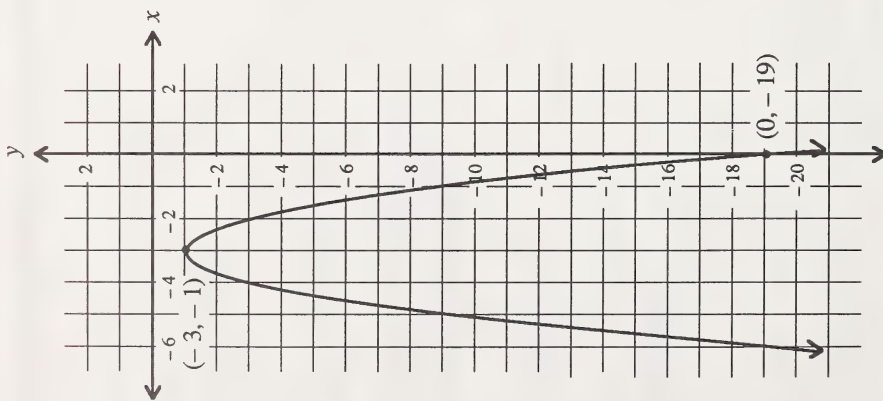
$$y = -2(9) - 1$$

$$y = -18 - 1$$

$$y = -19$$

The y -intercept is -19 .

vii.



b. $y = 2x^2 + 6x - 3$

$$y = 2\left(x^2 + 3x\right) - 3$$

$$y = 2\left(x^2 + 3x + \frac{9}{4}\right) - 3 - 2\left(\frac{9}{4}\right)$$

$$y = 2\left(x + \frac{3}{2}\right)^2 - 3 - \frac{9}{2}$$

$$y = 2\left(x + \frac{3}{2}\right)^2 - \frac{15}{2}$$

i. $\left(-\frac{3}{2}, -\frac{15}{2}\right)$

ii. The function has a minimum value of $-\frac{15}{2}$.

iii. $x = -\frac{3}{2}$

iv. $x \in R$

v. $y \geq -\frac{15}{2}$

vi. Let $x = 0$.

$$y = 2(0)^2 + 6(0) - 3$$

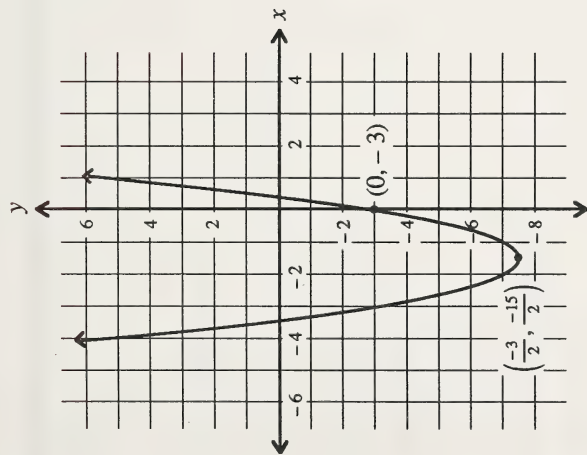
$$y = 2(0) + 0 - 3$$

$$y = 0 + 0 - 3$$

$$y = -3$$

The y-intercept is -3 .

vii.



vi. Let $x = 0$.

$$y = 3(0 - 1)^2 + 2$$

$$y = 3(-1)^2 + 2$$

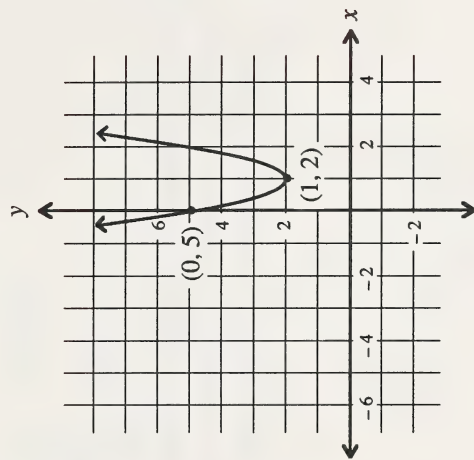
$$y = 3(1) + 2$$

$$y = 3 + 2$$

$$y = 5$$

The y-intercept is 5.

vii.



c. $y = 3(x - 1)^2 + 2$

i. $(1, 2)$

ii. The function has a minimum value of 2.

iii. $x = 1$

iv. $x \in \mathbb{R}$

v. $y \geq 2$

d. $y = -3x^2 + 6x - 2$

$$y = -3(x^2 - 2x) - 2$$

$$y = -3(x^2 - 2x + 1) - 2 - (-3)(1)$$

$$y = -3(x-1)^2 - 2 + 3$$

$$y = -3(x-1)^2 + 1$$

i. $(1, 1)$

ii. The function has a maximum value of 1.

iii. $x = 1$

iv. $x \in R$

v. $y \leq 1$

vi. Let $x = 0$.

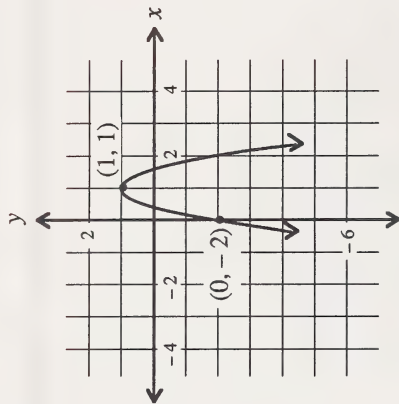
$$y = -3(0)^2 + 6(0) - 2$$

$$y = 0 + 0 - 2$$

$$y = -2$$

The y -intercept is -2 .

vii.



16. a. i. $(-2, 3)$

ii. The function has a maximum value of 3.

iii. $x = -2$

iv. $x \in R$

v. $y \leq 3$

vi. The y -intercept is -1 .

vii. $a = -1$

$$y = -(x+2)^2 + 3$$

b. i. $(0, 1)$

ii. The function has a minimum value of 1.

iii. $x = 0$

iv. $x \in R$

v. $y \geq 1$

vi. The y -intercept is 1.

vii. $a = 2$

$$y = 2(x-0)^2 + 1 \text{ or } y = 2x^2 + 1$$

c. i. $(-1, 0)$

ii. The function has a maximum value of 0.

iii. $x = -1$

iv. $x \in R$

v. $y \leq 0$

vi. The y-intercept is -3 .

vii. $a = -3$

$$y = -3(x+1)^2 + 0 \quad \text{or} \quad y = -3(x+1)^2$$

17. Put into completed square form.

$$h = -5t^2 + 60t$$

$$h = -5(t^2 - 12t \quad)$$

$$h = -5(t^2 - 12t + 36) - (-5)(36)$$

$$h = -5(t-6)^2 + 5(36)$$

$$h = -5(t-6)^2 + 180$$

a. The baseball reaches a maximum height of 180 m.

b. The baseball takes 6 s to reach this height.

c. The baseball is airborne for 12 s.

18. Since the width and the length of the rectangular region are unknown, you must define them in terms of a variable.

Let w equal the width of the rectangle.

For any rectangle the perimeter $= 2l + 2w$.

Therefore, $2l + 2w = 1000$

$$2l = 1000 - 2w$$

$$l = \frac{1000 - 2w}{2}$$

$$l = 500 - w$$

The length is $500 - w$.

Since you are required to find maximum area, you must set up an equation involving the area of a rectangle. You know that the area of a rectangle can be found by the formula $A = l \times w$.

For this situation, the length is $500 - w$ and the width equals w .

Therefore, $A = (500 - w)w$

$$A = 500w - w^2$$

$$A = -w^2 + 500w$$

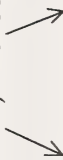
This is a quadratic function.

$$A = -1(w^2 - 500w)$$

) Put into completed square form.

$$A = -1(w^2 - 500w + 62\,500) - (-1)(62\,500)$$

$$A = -1(w - 250)^2 + 62\,500$$



This is the value for the width that gives the maximum area.

This value is the maximum area.

The maximum area enclosed is $62\,500 \text{ m}^2$.

The width of the field is 250 m and the length of the field is $500 - 250 = 250 \text{ m}$.

19. Define the numbers in terms of a variable.

Let n equal one of the numbers.

Therefore, $48 - n$ must be the other number.

Let S equal the sum of their squares.

$$S = n^2 + (48 - n)^2$$

$$S = n^2 + 2304 - 96n + n^2$$

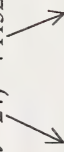
$$S = 2n^2 - 96n + 2304$$

$$S = 2(n^2 - 48n) + 2304$$

$$S = 2(n^2 - 48n + 576) + 2304 - 2(576)$$

$$S = 2(n - 24)^2 + 2304 - 1152$$

$$S = 2(n - 24)^2 + 1152$$



This value is the value for n that gives a minimum sum.

This value is the minimum sum.

One number is n and the other number is $48 - n$.
If $n = 24$, then $48 - n = 48 - 24 = 24$.

The two numbers are 24 and 24.

Extra Help

$$1. \quad y = -3x^2 + 2x - 1$$

$$a = -3, b = 2, c = -1$$

$$\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a} \right) = \left(\frac{-2}{2(-3)}, \frac{4(-3)(-1) - 2^2}{4(-3)} \right)$$

$$= \left(\frac{-2}{-6}, \frac{12 - 4}{-12} \right)$$

$$= \left(\frac{1}{3}, \frac{8}{-12} \right)$$

$$= \left(\frac{1}{3}, -\frac{2}{3} \right)$$

$$2. \quad y = 5x^2 - 15x$$

$$a = 5, b = -15, c = 0$$

$$\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a} \right) = \left(\frac{15}{2(5)}, \frac{4(5)(0) - (-15)^2}{4(5)} \right)$$

$$= \left(\frac{15}{10}, \frac{0 - 225}{20} \right)$$

$$= \left(\frac{3}{2}, -\frac{45}{4} \right)$$

$$3. \quad y = 4x^2 + x - 3$$

$$a = 4, b = 1, c = -3$$

$$\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a} \right) = \left(\frac{-1}{2(4)}, \frac{4(4)(-3) - 1^2}{4(4)} \right)$$

$$= \left(\frac{-1}{8}, \frac{-48 - 1}{16} \right)$$

$$= \left(\frac{-1}{8}, -\frac{49}{16} \right)$$

$$4. \quad y = 2x^2 + 6$$

$$a = 2, b = 0, c = 6$$

$$\left(\frac{-b}{2a}, \frac{4ac - b^2}{4a} \right) = \left(\frac{-0}{2(2)}, \frac{4(2)(6) - 0^2}{4(2)} \right)$$

$$= \left(\frac{0}{4}, \frac{48 - 0}{8} \right)$$

$$= (0, 6)$$

5. $y - 3 + 2x^2 = -6x$

Put in standard form.

$$y = -2x^2 - 6x + 3$$

$$a = -2, b = -6, c = 3$$

$$\begin{aligned} \left(\frac{-b}{2a}, \frac{4ac - b^2}{4a} \right) &= \left(\frac{-(-6)}{2(-2)}, \frac{4(-2)(3) - (-6)^2}{4(-2)} \right) \\ &= \left(\frac{6}{-4}, \frac{-24 - 36}{-8} \right) \\ &= \left(-\frac{3}{2}, \frac{-60}{-8} \right) \\ &= \left(-\frac{3}{2}, \frac{15}{2} \right) \end{aligned}$$

6. $y + 3 = 4x + 3x^2$

Put in standard form.

$$y = 3x^2 + 4x - 3$$

$$a = 3, b = 4, c = -3$$

$$\begin{aligned} \left(\frac{-b}{2a}, \frac{4ac - b^2}{4a} \right) &= \left(\frac{-4}{2(3)}, \frac{4(3)(-3) - 4^2}{4(3)} \right) \\ &= \left(\frac{-4}{6}, \frac{-36 - 16}{12} \right) \\ &= \left(-\frac{2}{3}, -\frac{52}{12} \right) \\ &= \left(-\frac{2}{3}, -\frac{13}{3} \right) \end{aligned}$$

Extensions

$$y = ax^2 + bx + c$$

(Put into completed square form.)

$$y = (ax^2 + bx) + c$$

(Put parentheses around the first two terms.)

$$y = a \left(x^2 + \frac{b}{a}x \right) + c$$

(Factor a from the first two terms.)

$$y = a \left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} \right) + c - a \left(\frac{b^2}{4a^2} \right)$$

(Complete the square.)

$$\begin{aligned} \left(\frac{1}{2} \times \frac{b}{a} \right)^2 &= \left(\frac{b}{2a} \right)^2 \\ &= \frac{b^2}{4a^2} \end{aligned}$$

(Factor.)

$$y = a \left(x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a}$$

$$y = a \left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a}$$

(Simplify.)

$$y = a \left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a}$$

(This is in completed square form.)

The maximum or minimum occurs at $\frac{4ac - b^2}{4a}$.

The equation for the axis of symmetry is $x = -\frac{b}{2a}$.



Exploring Topic 4

Activity 1

Find the x-intercepts for a quadratic function by solving the corresponding quadratic equation.

1. The y-intercept for a function is the point where the graph crosses the y-axis. The value of x at this point is zero.
2. The x-intercept for a function is the point where the graph crosses the x-axis. The value of y at this point is zero.

3. $y = x^2 - 2x - 8$

y-intercept
Let $x = 0$.

$$y = (0)^2 - 2(0) - 8$$

$$y = 0 - 0 - 8$$

$$y = -8$$

The y-intercept is -8 .

x-intercepts
Let $y = 0$.

$$0 = x^2 - 2x - 8$$

$$0 = (x - 4)(x + 2)$$

$$x - 4 = 0 \text{ or } x + 2 = 0$$

$$x = 4 \quad x = -2$$

The x-intercepts are 4 and -2 .

4. The following is a quadratic function:

$$y = ax^2 + bx + c$$

The following is a quadratic equation:

$$0 = ax^2 + bx + c$$

5. If -7 is a root, then -7 will satisfy the equation $x^2 + 4x - 21 = 0$.

$$(-7)^2 + 4(-7) - 21 = 0$$

$$49 - 28 - 21 = 0$$

$$0 = 0$$

The root -7 satisfies the equation.

6. a. $2x^2 - 3x + 1 = 0$

b. $7x^2 + 3x = 0$

c. $2x^2 + 7 = 0$

7. a. $x^2 - 25 = 0$

$$(x + 5)(x - 5) = 0$$

$$x + 5 = 0 \text{ or } x - 5 = 0$$

$$x = -5 \quad x = 5$$

The roots are -5 and 5 .

b. $x^2 = 1$

$$x^2 - 1 = 0$$

$$(x+1)(x-1) = 0$$

$$x+1=0 \text{ or } x-1=0$$

$$x = -1 \quad x = 1$$

The roots are -1 and 1 .

c. $x^2 - 3 = 0$

$$x^2 = 3$$

$$x = \pm\sqrt{3}$$

The roots are $\sqrt{3}$ and $-\sqrt{3}$.

d. $x^2 + 4 = 0$

$$x^2 = -4$$

$$x = \pm\sqrt{-4}$$

There are no real roots.

e. $x^2 = -25$

$$x = \pm\sqrt{-25}$$

There are no real roots.

f. $4x^2 - 1 = 0$

$$(2x+1)(2x-1) = 0$$

$$2x+1=0 \text{ or } 2x-1=0$$

$$2x = -1 \quad 2x = 1$$

$$x = -\frac{1}{2} \quad x = \frac{1}{2}$$

The roots are $-\frac{1}{2}$ and $\frac{1}{2}$.

g. $3x^2 - 27 = 0$

$$3(x^2 - 9) = 0$$

$$3(x+3)(x-3) = 0$$

$$x+3=0 \text{ or } x-3=0$$

$$x = -3 \quad x = 3$$

The roots are -3 and 3 .

h. $3x^2 = 0$

$$x^2 = \frac{0}{3}$$

$$x^2 = 0$$

$$x = \pm\sqrt{0}$$

$$x = 0$$

The root is 0 .

i. $x^2 - 2 = 0$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

The roots are $\sqrt{2}$ and $-\sqrt{2}$.

j. $-2x^2 - 8 = 0$

$$-2(x^2 + 4) = 0$$

$$\frac{-2(x^2 + 4)}{-2} = \frac{0}{-2}$$

$$x^2 + 4 = 0$$

$$x^2 = -4$$

$$x = \pm\sqrt{-4}$$

There are no real roots.

8. a. $3x^2 - 12x = 0$

$$3x(x - 4) = 0$$

$$3x = 0 \text{ or } x - 4 = 0$$

$$x = 0 \quad x = 4$$

The roots are 0 and 4.

b. $x^2 + 7x = 0$

$$x(x + 7) = 0$$

$$x = 0 \text{ or } x + 7 = 0$$

$$x = -7$$

The roots are 0 and -7 .

c. $-x^2 + 5x = 0$

$$-x(x - 5) = 0$$

$$-x = 0 \text{ or } x - 5 = 0$$

$$x = 0 \quad x = 5$$

The roots are 0 and 5.

d. $x^2 = 4x$

$$x^2 - 4x = 0$$

$$x(x - 4) = 0$$

$$x = 0 \text{ or } x - 4 = 0$$

$$x = 4$$

The roots are 0 and 4.

e. $+5x = -x^2$

$$x^2 + 5x = 0$$

$$x(x+5) = 0$$

$$x = 0 \text{ or } x+5 = 0$$

$$x = -5$$

The roots are 0 and -5 .

f. $2x^2 = 16x$

$$2x^2 - 16x = 0$$

$$2x(x-8) = 0$$

$$2x = 0 \text{ or } x-8 = 0$$

$$x = 0 \quad x = 8$$

The roots are 0 and 8.

g. $-3x^2 - 7x = 0$

$$-x(3x+7) = 0$$

$$-x = 0 \text{ or } 3x+7 = 0$$

$$x = 0 \quad 3x = -7$$

$$x = -\frac{7}{3}$$

The roots are 0 and $-\frac{7}{3}$.

h. $2x^2 + 5x = 0$

$$x(2x+5) = 0$$

$$x = 0 \text{ or } 2x+5 = 0$$

$$2x = -5$$

$$x = -\frac{5}{2}$$

The roots are 0 and $-\frac{5}{2}$.

9. a. $x^2 - 3x + 2 = 0$

$$(x-2)(x-1) = 0$$

$$x-2 = 0 \text{ or } x-1 = 0$$

$$x = 2 \quad x = 1$$

The roots are 2 and 1.

b. $x^2 = 2x-1$

$$x^2 - 2x + 1 = 0$$

$$(x-1)(x-1) = 0$$

$$x-1 = 0 \text{ or } x-1 = 0$$

$$x = 1 \quad x = 1$$

The root is 1.

c. $0 = x^2 - 7x + 12$

$$0 = (x-3)(x-4)$$

$$x-3=0 \text{ or } x-4=0$$

$$x=3 \quad x=4$$

The roots are 3 and 4.

d. $x^2 - x - 56 = 0$

$$(x-8)(x+7) = 0$$

$$x-8=0 \text{ or } x+7=0$$

$$x=8 \quad x=-7$$

The roots are 8 and -7.

e. $x^2 - 3x = 28$

$$x^2 - 3x - 28 = 0$$

$$(x-7)(x+4) = 0$$

$$x-7=0 \text{ or } x+4=0$$

$$x=7 \quad x=-4$$

The roots are 7 and -4.

f. $0 = x^2 - 5x - 14$

$$0 = (x-7)(x+2)$$

$$x-7=0 \text{ or } x+2=0$$

$$x=7 \quad x=-2$$

The roots are 7 and -2.

g. $x^2 + 3x - 4 = 0$

$$(x+4)(x-1) = 0$$

$$x+4=0 \text{ or } x-1=0$$

$$x=-4 \quad x=1$$

The roots are -4 and 1.

h. $-x^2 - x = -20$

$$-x^2 - x + 20 = 0$$

$$-1(-x^2 - x + 20) = -1(0)$$

$$x^2 + x - 20 = 0$$

$$(x+5)(x-4) = 0$$

$$x+5=0 \text{ or } x-4=0$$

$$x=-5 \quad x=4$$

The roots are -5 and 4.

i. $x^2 + 3x + 5 = 0$

This equation is not factorable.

The roots cannot be determined by factoring.

10. a. $4x^2 - 4x + 1 = 0$

$$(2x-1)(2x-1) = 0$$

$$2x-1=0 \text{ or } 2x-1=0$$

$$2x=1 \quad 2x=1$$

$$x = \frac{1}{2} \quad x = \frac{1}{2}$$

The root is $\frac{1}{2}$.

b. $2x^2 + 7x + 6 = 0$

$$(2x+3)(x+2) = 0$$

$$2x+3=0 \text{ or } x+2=0$$

$$2x=-3 \quad x=-2$$

$$x = -\frac{3}{2}$$

The roots are $-\frac{3}{2}$ and -2 .

c. $2x^2 + 11x = 6$

$$2x^2 + 11x - 6 = 0$$

$$(2x-1)(x+6) = 0$$

$$2x-1=0 \text{ or } x+6=0$$

$$2x=1 \quad x=-6$$

$$x = \frac{1}{2}$$

The roots are $\frac{1}{2}$ and -6 .

d. $6x^2 + 23x + 20 = 0$

$$(3x+4)(2x+5) = 0$$

$$3x+4=0 \text{ or } 2x+5=0$$

$$3x=-4 \quad 2x=-5$$

$$x = -\frac{4}{3} \quad x = -\frac{5}{2}$$

The roots are $-\frac{4}{3}$ and $-\frac{5}{2}$.

e. $4x^2 - 7x + 3 = 0$

$$(4x-3)(x-1) = 0$$

$$4x-3=0 \text{ or } x-1=0$$

$$4x=3 \quad x=1$$

$$x = \frac{3}{4}$$

The roots are $\frac{3}{4}$ and 1 .

f. $5x^2 - 7x - 6 = 0$

$$(5x+3)(x-2) = 0$$

$$5x+3=0 \text{ or } x-2=0$$

$$5x=-3 \quad x=2$$

$$x = -\frac{3}{5}$$

The roots are $-\frac{3}{5}$ and 2 .

g. $7x^2 - 2x - 5 = 0$

$$(7x + 5)(x - 1) = 0$$

$$7x + 5 = 0 \text{ or } x - 1 = 0$$

$$7x = -5 \quad x = 1$$

$$x = -\frac{5}{7}$$

The roots are $-\frac{5}{7}$ and 1.

h. $5x^2 - 2x - 7 = 0$

$$(5x - 7)(x + 1) = 0$$

$$5x - 7 = 0 \text{ or } x + 1 = 0$$

$$5x = 7 \quad x = -1$$

$$x = \frac{7}{5}$$

The roots are $\frac{7}{5}$ and -1 .

i. $-5x^2 = -3 + 14x$

$$-5x^2 - 14x + 3 = 0$$

$$-1(-5x^2 - 14x + 3) = -1(0)$$

$$5x^2 + 14x - 3 = 0$$

$$(5x - 1)(x + 3) = 0$$

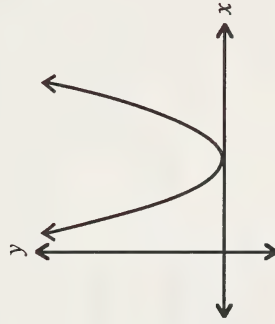
$$5x - 1 = 0 \text{ or } x + 3 = 0$$

$$5x = 1 \quad x = -3$$

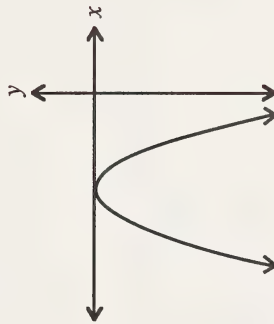
$$x = \frac{1}{5}$$

The roots are $\frac{1}{5}$ and -3 .

11.



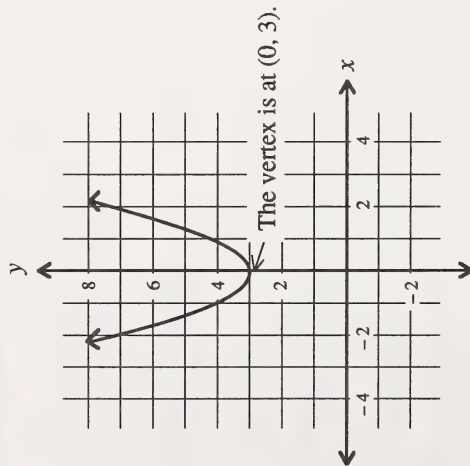
or



If there is only one x-intercept, the parabola would touch the x-axis at one point. The parabola could either open upward or downward.

12. If a quadratic equation does not have any real roots, then the graph of the corresponding function will not have any x -intercepts. The graph will be entirely above the x -axis or entirely below the x -axis.

13. The graph of $y = x^2 + 3$ is congruent to the graph of $y = x^2$ translated three units upward. The vertex is at $(0, 3)$. Since the graph opens upward, no x -intercepts can exist.



14. a. $y = 3x^2 - 3$
Let $y = 0$.

$$0 = 3x^2 - 3$$

$$0 = 3(x^2 - 1)$$

$$0 = 3(x+1)(x-1)$$

$$x+1=0 \text{ or } x-1=0$$

$$x = -1 \quad x = 1$$

The x -intercepts are -1 and 1 .

b. $y = 3x^2 - 3x$

Let $y = 0$.

$$0 = 3x^2 - 3x$$

$$0 = 3x(x-1)$$

$$3x=0 \text{ or } x-1=0$$

$$x=0 \quad x=1$$

The x -intercepts are 0 and 1 .

c. $y = x^2 - 3x - 4$

Let $y = 0$.

$$0 = x^2 - 3x - 4$$

$$0 = (x-4)(x+1)$$

$$x-4=0 \text{ or } x+1=0$$

$$x=4 \quad x=-1$$

The x -intercepts are 4 and -1 .

d. $y = 2x^2 - 3x - 5$

Let $y = 0$.

$$0 = 2x^2 - 3x - 5$$

$$0 = (2x - 5)(x + 1)$$

$$2x - 5 = 0 \text{ or } x + 1 = 0$$

$$2x = 5 \quad x = -1$$

$$x = \frac{5}{2}$$

The x-intercepts are $\frac{5}{2}$ and -1 .

e. $y = x^2 + 9$

Let $y = 0$.

$$0 = x^2 + 9$$

$$-9 = x^2$$

$$\pm\sqrt{-9} = x$$

Since $\sqrt{-9}$ is not a real number, there are no x-intercepts.

f. $y = -x^2 - 12x$

Let $y = 0$.

$$0 = -x^2 - 12x$$

$$0 = -x(x + 12)$$

$$-x = 0 \text{ or } x + 12 = 0$$

$$x = 0 \quad x = -12$$

The x-intercepts are 0 and -12 .

g. $y = x^2 + x - 30$

Let $y = 0$.

$$0 = x^2 + x - 30$$

$$0 = (x + 6)(x - 5)$$

$$x + 6 = 0 \text{ or } x - 5 = 0$$

$$x = -6 \quad x = 5$$

The x-intercepts are -6 and 5 .

h. $y = x^2 - 8x + 16$

Let $y = 0$.

$$0 = x^2 - 8x + 16$$

$$0 = (x - 4)(x - 4)$$

$$x - 4 = 0 \text{ or } x - 4 = 0$$

$$x = 4 \quad x = 4$$

The x-intercept is 4.

Extra Help

1. a. $x^2 - 5x - 24 = (x - 8)(x + 3)$

b. $x^2 - x - 30 = (x - 6)(x + 5)$

c. $x^2 - 2x + 1 = (x - 1)(x - 1)$ or $(x - 1)^2$

d. $x^2 - 3x - 40 = (x - 8)(x + 5)$

e. $x^2 + x - 20 = (x+5)(x-4)$

f. $x^2 + 9x + 14 = (x+2)(x+7)$

g. $x^2 - 5x - 6 = (x-6)(x+1)$

h. $x^2 + 2x - 8 = (x+4)(x-2)$

i. $x^2 + 3x - 8$
not factorable

2. a. $6x^2 + 13x + 6$

$$(6)(6) = 36$$

$$(4)(9) = 36$$

$$4 + 9 = 13$$

$$\begin{aligned} 6x^2 + 4x + 9x + 6 &= (6x^2 + 4x) + (9x + 6) \\ &= 2x(3x + 2) + 3(3x + 2) \\ &= (3x + 2)(2x + 3) \end{aligned}$$

b. $6x^2 - 13x + 6$

$$(6)(6) = 36$$

$$(-4)(-9) = 36$$

$$(-4) + (-9) = -13$$

$$\begin{aligned} 6x^2 - 4x + (-9x) + 6 &= (6x^2 - 4x) + (-9x + 6) \\ &= 2x(3x - 2) - 3(3x - 2) \\ &= (3x - 2)(2x - 3) \end{aligned}$$

c. $6x^2 - 5x - 6$

$$(6)(-6) = -36$$

$$(-9)(4) = -36$$

$$(-9) + 4 = -5$$

$$\begin{aligned} 6x^2 - 9x + 4x - 6 &= (6x^2 - 9x) + (4x - 6) \\ &= 3x(2x - 3) + 2(2x - 3) \\ &= (2x - 3)(3x + 2) \end{aligned}$$

d. $6x^2 + 5x - 6$

$$(6)(-6) = -36$$

$$(9)(-4) = -36$$

$$9 + (-4) = 5$$

$$\begin{aligned} 6x^2 + 9x + (-4x) - 6 &= (6x^2 + 9x) + (-4x - 6) \\ &= 3x(2x + 3) - 2(2x + 3) \\ &= (2x + 3)(3x - 2) \end{aligned}$$

e. $6x^2 + 11x + 5$

$$(6)(5) = 30$$

$$6 + 5 = 11$$

$$\begin{aligned} (6x^2 + 6x) + (5x + 5) &= 6x(x + 1) + 5(x + 1) \\ &= (x + 1)(6x + 5) \end{aligned}$$

f. $6x^2 - 17x + 5$

$$(6)(5) = 30$$

$$(-15)(-2) = 30$$

$$(-15) + (-2) = -17$$

$$\begin{aligned} (6x^2 - 15x) + (-2x + 5) &= 3x(2x - 5) - 1(2x - 5) \\ &= (2x - 5)(3x - 1) \end{aligned}$$

g. $6x^2 + x - 5$

$$(6)(-5) = -30$$

$$6 + (-5) = 1$$

$$\begin{aligned} (6x^2 + 6x) + (-5x - 5) &= 6x(x + 1) - 5(x + 1) \\ &= (x + 1)(6x - 5) \end{aligned}$$

h. $6x^2 - 29x - 5$

$$(6)(-5) = -30$$

$$(-30)(1) = -30$$

$$(-30) + 1 = -29$$

$$\begin{aligned}(6x^2 - 30x) + (1x - 5) &= 6x(x - 5) + 1(x - 5) \\ &= (x - 5)(6x + 1)\end{aligned}$$

i. $6x^2 - 2x - 5$

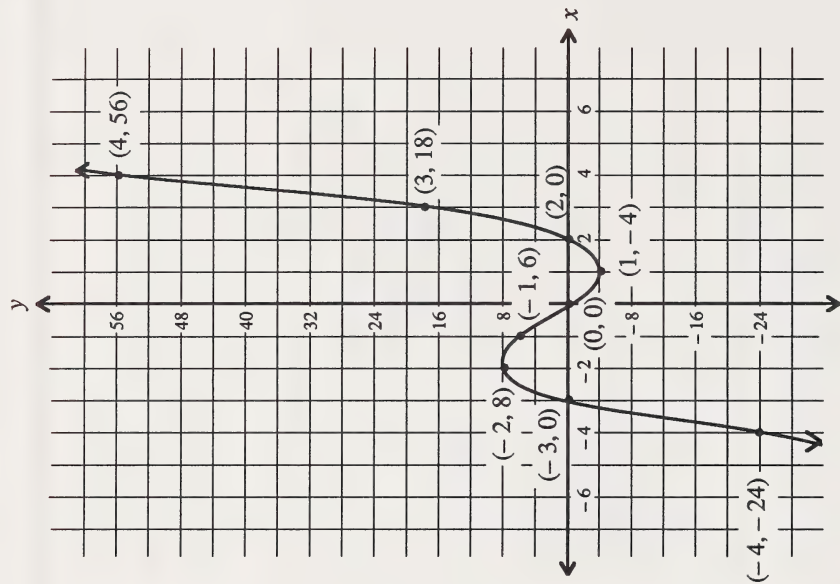
$$(6)(-5) = -30$$

This trinomial is not factorable because you cannot find two numbers whose product is -30 and whose sum is -2 .

Extensions

$$y = x^3 + x^2 - 6x$$

x	-4	-3	-2	-1	0	1	2	3	4
y	-24	0	8	6	0	-4	0	18	56



The x -intercepts are at 0, -3 , and 2.

The corresponding equation is $x^3 + x^2 - 6x = 0$.
Factor.

$$x(x^2 + x - 6) = 0$$

$$x(x+3)(x-2) = 0$$

$$x = 0 \text{ or } x+3=0 \text{ or } x-2=0$$

$$x = -3 \quad x = 2$$

The roots are 0, -3, and 2.

The roots and the x -intercepts have the same values.



Exploring Topic 5

Activity 1

Solve quadratic equations by using the quadratic formula.

1. a. $2x^2 - 3x + 3 = 0$
 $a = 2 \quad b = -3 \quad c = 3$

b. $-2x^2 - 7 = 0$
 $a = -2 \quad b = 0 \quad c = -7$

c. $3x^2 - 4x = 0$
 $a = 3 \quad b = -4 \quad c = 0$

2. a. $x^2 = 9$

$$x = \pm\sqrt{9}$$

$$x = \pm 3$$

b. $x^2 = 5$

$$x = \pm\sqrt{5}$$

c. $x^2 = -4$

$$x = \pm\sqrt{-4}$$

Since $\pm\sqrt{-4}$ is not a real number, there are no real roots.

d. $(x-2)^2 = 9$

$$x-2 = \pm\sqrt{9}$$

$$x-2 = \pm 3$$

$$x = 2 \pm 3$$

$$x = 5 \text{ or } x = -1$$

e. $(x+2)^2 = 25$

$$x+2 = \pm\sqrt{25}$$

$$x+2 = \pm 5$$

$$x = -2 \pm 5$$

$$x = -7 \text{ or } x = 3$$

f. $(x-3)^2 = 7$

$$x-3 = \pm\sqrt{7}$$

$$x = 3 \pm \sqrt{7}$$

g. $(2x+3)^2 = 4$

$$2x+3 = \pm\sqrt{4}$$

$$2x+3 = \pm 2$$

$$2x = -3 \pm 2$$

$$x = \frac{-3 \pm 2}{2}$$

$$x = -\frac{1}{2} \text{ or } x = -\frac{5}{2}$$

h. $(3x-7)^2 = 5$

$$3x-7 = \pm\sqrt{5}$$

$$3x = 7 \pm \sqrt{5}$$

$$x = \frac{7 \pm \sqrt{5}}{3}$$

3. $x^2 - 4x + 2 = 0$

Verify $2 + \sqrt{2}$.

LS	RS
$(2 + \sqrt{2})^2 - 4(2 + \sqrt{2}) + 2$	0
$4 + 4\sqrt{2} + 2 - 8 - 4\sqrt{2} + 2$	0
$(4 + 2 - 8 + 2) + (4\sqrt{2} - 4\sqrt{2})$	0
0 + 0	0
0	0
LS	RS

Thus, $2 + \sqrt{2}$ is a root.

Verify $2 - \sqrt{2}$.

LS	RS
$(2 - \sqrt{2})^2 - 4(2 - \sqrt{2}) + 2$	0
$4 - 4\sqrt{2} + 2 - 8 + 4\sqrt{2} + 2$	0
$(4 + 2 - 8 + 2) + (-4\sqrt{2} + 4\sqrt{2})$	0
0 + 0	0
0	0
LS	RS

Thus, $2 - \sqrt{2}$ is a root.

4. $\sqrt{7} \div 2.645751311$ (Use your calculator.)

$$\frac{5 + \sqrt{7}}{2} \div \frac{5 + 2.645751311}{2}$$

$$\div \frac{7.645751311}{2}$$

$$\div 3.822875656$$

$$\div 3.8$$

$$\frac{5 - \sqrt{7}}{2} \div \frac{5 - 2.645751311}{2}$$

$$\div \frac{2.354248689}{2}$$

$$\div 1.177124345$$

$$\div 1.2$$

5. a. $3x^2 - 6x = 3(x^2 - 2x)$ b. $3x^2 - 2x = 3\left(x^2 - \frac{2}{3}x\right)$

c. $3x^2 + 5x = 3\left(x^2 + \frac{5}{3}x\right)$

6. The quadratic formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The formula should be used to find the roots of a quadratic equation that is not readily factorable.

7. a. $x^2 - 7x + 6 = 0$ (Factor.)

$$(x-6)(x-1) = 0$$

$$x-6=0 \text{ or } x-1=0$$

$$x=6 \quad x=1$$

The roots are 6 and 1.

b. $-2x^2 - 5x + 4 = 0$ (not factorable)

$$a = -2, b = -5, c = 4$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(-2)(4)}}{2(-2)}$$

$$x = \frac{5 \pm \sqrt{25 + 32}}{-4}$$

$$x = \frac{5 \pm \sqrt{57}}{-4}$$

$$x = \frac{-5 \pm \sqrt{57}}{4}$$

The roots are $x = \frac{-5 + \sqrt{57}}{4}$ and $x = \frac{-5 - \sqrt{57}}{4}$.

c. $x^2 - 144 = 0$ (Factor.)

$$(x+12)(x-12) = 0$$

$$x+12=0 \text{ or } x-12=0$$

$$x=-12 \quad x=12$$

The roots are 12 and -12.

Another method is as follows:

$$x^2 - 144 = 0$$

$$x^2 = 144$$

$$\sqrt{x^2} = \pm\sqrt{144}$$

$$x = \pm 12$$

The roots are 12 and -12.

d. $3x^2 - 2x - 3 = 0$ (not factorable)

$$a = 3, b = -2, c = -3$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-3)}}{2(3)}$$

$$x = \frac{2 \pm \sqrt{4 + 36}}{6}$$

$$x = \frac{2 \pm \sqrt{40}}{6}$$

$$x = \frac{2 \pm 2\sqrt{10}}{6}$$

$$x = \frac{\cancel{2}(1 \pm \sqrt{10})}{\cancel{6}_3}$$

$$x = \frac{1 \pm \sqrt{10}}{3}$$

The roots are $\frac{1+\sqrt{10}}{3}$ and $\frac{1-\sqrt{10}}{3}$.

e. $x^2 - 4x + 6 = 0$ (not factorable)

$$a = 1, b = -4, c = 6$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(6)}}{2(1)}$$

$$x = \frac{4 \pm \sqrt{16 - 24}}{2}$$

$$x = \frac{4 \pm \sqrt{-8}}{2}$$

Since $\sqrt{-8}$ is not a real number, there are no real roots.

f. $3x^2 - 12x = 0$ (Factor.)

$$3x(x - 4) = 0$$

$$3x = 0 \text{ or } x - 4 = 0$$

$$x = \frac{0}{3} \quad x = 4$$

$$x = 0$$

The roots are 0 and 4.

g. $16x^2 - 2x + 1 = 0$
 $a = 16, b = -2, c = 1$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(16)(1)}}{2(16)}$$

$$x = \frac{2 \pm \sqrt{4 - 64}}{32}$$

$$x = \frac{2 \pm \sqrt{-60}}{32}$$

Since $\sqrt{-60}$ is not a real number, there are no real roots.

h. $x^2 - x - 1 = 0$
 $a = 1, b = -1, c = -1$
(not factorable)

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{1 + 4}}{2}$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

The roots are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.

i. $x^2 - \frac{1}{2}x - \frac{3}{4} = 0$

$$4\left(x^2 - \frac{1}{2}x - \frac{3}{4}\right) = 4(0)$$

$$4x^2 - 2x - 3 = 0 \quad (\text{not factorable})$$

$$a = 4, b = -2, c = -3$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(4)(-3)}}{2(4)}$$

$$x = \frac{2 \pm \sqrt{4 + 48}}{8}$$

$$x = \frac{2 \pm \sqrt{52}}{8}$$

$$x = \frac{2 \pm 2\sqrt{13}}{8}$$

$$x = \frac{1 \pm \sqrt{13}}{2}$$

$$x = \frac{1 \pm \sqrt{13}}{4}$$

The roots are $\frac{1+\sqrt{13}}{4}$ and $\frac{1-\sqrt{13}}{4}$.

8. To find x -intercepts, let $y = 0$.

$$0 = 5x^2 - 4x - 3 \quad (\text{not factorable})$$

$$a = 5, b = -4, c = -3$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(5)(-3)}}{2(5)}$$

$$x = \frac{4 \pm \sqrt{16 + 60}}{10}$$

$$x = \frac{4 \pm \sqrt{76}}{10}$$

$$x = \frac{4 \pm 2\sqrt{19}}{10}$$

$$x = \frac{2(2 \pm \sqrt{19})}{2(5)}$$

$$x = \frac{2 \pm \sqrt{19}}{5}$$

$$x \doteq \frac{2 \pm 4.358898944}{5}$$

$$x \doteq \frac{6.358898944}{5} \quad \text{or} \quad x \doteq \frac{-2.358898944}{5}$$

$$x \doteq 1.271779789 \quad x \doteq -0.471779788$$

$$x \doteq 1.27 \quad x \doteq -0.47$$

The x -intercepts are approximately 1.27 and -0.47 .

9. a. When $h = 340$ m,

$$340 = -5t^2 + 100t + 20$$

$$5t^2 - 100t - 20 + 340 = 0$$

$$5t^2 - 100t + 320 = 0 \quad (\text{Divide both sides by 5.})$$

$$t^2 - 20t + 64 = 0 \quad (\text{Factor.})$$

$$(t - 16)(t - 4) = 0$$

$$t - 16 = 0 \quad \text{or} \quad t - 4 = 0$$

$$t = 16 \quad t = 4$$

The projectile has a height of 340 m after 4 s and 16 s.

- b. When $h = 250$ m,

$$250 = -5t^2 + 100t + 20$$

$$5t^2 - 100t - 20 + 250 = 0$$

$$5t^2 - 100t + 230 = 0 \quad (\text{Divide both sides by 5.})$$

$$t^2 - 20t + 46 = 0 \quad (\text{not factorable})$$

$$a = 1, b = -20, c = 46$$

$$t = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(1)(46)}}{2(1)}$$

$$t = \frac{20 \pm \sqrt{400 - 184}}{2}$$

$$t = \frac{20 \pm \sqrt{216}}{2}$$

$$t = \frac{20 \pm 2\sqrt{54}}{2}$$

$$t = \frac{2(10 \pm \sqrt{54})}{2}$$

$$t = 10 \pm \sqrt{54}$$

The projectile has a height of 250 m after $10 - \sqrt{54}$ s (or about 2.65 s) and after $10 + \sqrt{54}$ s (or about 17.35 s).

c. When t is $2\sqrt{3}$ s,

$$h = -5(2\sqrt{3})^2 + 100(2\sqrt{3}) + 20$$

$$h = -5(12) + 200\sqrt{3} + 20$$

$$h = -60 + 200\sqrt{3} + 20$$

$$h = -40 + 200\sqrt{3}$$

The height after $2\sqrt{3}$ s is $(-40 + 200\sqrt{3})$ m.

The decimal approximation is as follows:

$$\sqrt{3} \doteq 1.732\ 050\ 808$$

$$-40 + 200\sqrt{3} \doteq -40 + 200(1.732\ 050\ 808)$$

$$\doteq 306.410\ 161\ 5$$

$$\doteq 306.41$$

The height after $2\sqrt{3}$ s is about 306.41 m.

10. a. Define the unknowns in terms of a variable.

Let w equal the width of the field.

Therefore, $w + 40$ is equal to the length.

Since area is equal to length times width, you get the following:

$$2100 = (w + 40)w \quad (\text{Factor.})$$

$$2100 = w^2 + 40w$$

$$0 = w^2 + 40w - 2100$$

$$0 = (w + 70)(w - 30)$$

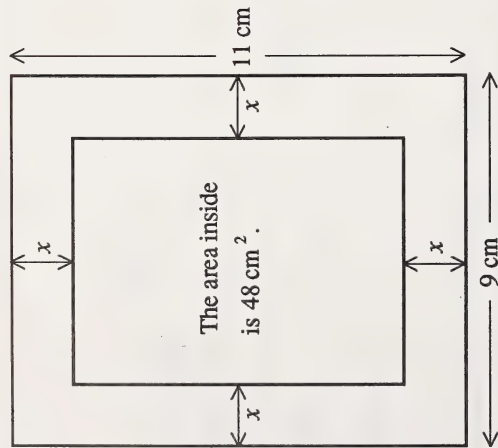
$$w + 70 = 0 \quad \text{or} \quad w - 30 = 0$$

$$w = -70 \quad w = 30$$

The root $w = -70$ is inadmissible since the width of a field cannot be a negative quantity.

The width of the field is 30 m, and since the length is represented by $w + 40$, it must be $30 + 40 = 70$ m.

b. Draw a diagram.



Let the width of the border be x .

The width of the region inside the border is $9 - 2x$.

The length of the region inside the border is $11 - 2x$.

Since area equals length times width, you get the following:

$$48 = (11 - 2x)(9 - 2x)$$

$$48 = 99 - 40x + 4x^2$$

$$0 = 4x^2 - 40x + 51 \quad (\text{not factorable})$$

$$a = 4, b = -40, c = 51$$

$$x = \frac{-(-40) \pm \sqrt{(-40)^2 - 4(4)(51)}}{2(4)}$$

$$x = \frac{40 \pm \sqrt{1600 - 816}}{8}$$

$$x = \frac{40 \pm \sqrt{784}}{8}$$

$$x = \frac{40 \pm 28}{8}$$

$$x = \frac{40 + 28}{8} \quad \text{or} \quad x = \frac{40 - 28}{8}$$

$$x = 8.5 \quad x = 1.5$$

The root 8.5 is inadmissible since the border cannot possibly be 8.5 cm.

Thus, the border should be 1.5 cm in width.

c. Let x be the first positive integer.

Therefore, $x + 1$ must be the next consecutive integer.

If the sum of their squares is 421, then you get the following equation:

$$x^2 + (x + 1)^2 = 421$$

$$x^2 + x^2 + 2x + 1 = 421$$

$$2x^2 + 2x - 420 = 0$$

$$x^2 + x - 210 = 0 \quad (\text{Factor.})$$

$$(x + 15)(x - 14) = 0$$

$$x + 15 = 0 \text{ or } x - 14 = 0$$

$$x = -15 \quad x = 14$$

Note that $x = -15$ is inadmissible since it is not positive.

The two positive integers are 14 and $14 + 1 = 15$.

- d. Let x be the first integer.

Therefore, $x + 2$ must be the next consecutive **even** integer.
If the sum of their squares is 452, then you get the following equation:

$$x^2 + (x + 2)^2 = 452$$

$$x^2 + x^2 + 4x + 4 = 452$$

$$2x^2 + 4x - 448 = 0$$

$$x^2 + 2x - 224 = 0$$

(Factor.)

$$(x + 16)(x - 14) = 0$$

$$x + 16 = 0 \text{ or } x - 14 = 0$$

$$x = -16 \quad x = 14$$

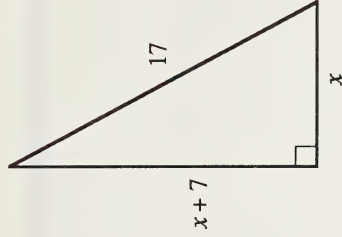
Since it is not positive, $x = -16$ is inadmissible.

The two positive integers are 14 and $14 + 2 = 16$.

- e. Draw a diagram.

Let the short side of the triangle equal x .

Therefore, the other side must be $x + 7$.



Use the Pythagorean theorem.

$$(17)^2 = x^2 + (x + 7)^2$$

$$289 = x^2 + x^2 + 14x + 49$$

$$0 = 2x^2 + 14x + 49 - 289$$

$$0 = 2x^2 + 14x - 240$$

$$0 = x^2 + 7x - 120$$

$$0 = (x + 15)(x - 8)$$

$$x + 15 = 0 \text{ or } x - 8 = 0$$

$$x = -15 \quad x = 8$$

Since a length cannot be negative, $x = -15$ is inadmissible.

The shorter side is 8 cm.

The other side is $8 + 7 = 15$ cm.

Extra Help

$$1. \quad x^2 - 5x + 2 = 0$$

$\downarrow$ $\swarrow$
 $a=1$ $b=-5$ $c=2$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{5 \pm \sqrt{25 - 8}}{2}$$

$$x = \frac{5 \pm \sqrt{17}}{2}$$

$$2. \quad x^2 + 3x - 7 = 0$$

$\downarrow$ $\swarrow$
 $a=1$ $b=3$ $c=-7$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-7)}}{2(1)}$$

$$x = \frac{-3 \pm \sqrt{9 + 28}}{2}$$

$$x = \frac{-3 \pm \sqrt{37}}{2}$$

$$3. \quad 3x^2 + 5x + 1 = 0$$

$\downarrow$ $\swarrow$
 $a=3$ $b=5$ $c=1$

$$x = \frac{-5 \pm \sqrt{(5)^2 - 4(3)(1)}}{2(3)}$$

$$x = \frac{-5 \pm \sqrt{25 - 12}}{6}$$

$$x = \frac{-5 \pm \sqrt{13}}{6}$$

$$4. \quad 2x^2 - 3x - 1 = 0$$

$\downarrow$ $\swarrow$
 $a=2$ $b=-3$ $c=-1$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)}$$

$$x = \frac{3 \pm \sqrt{9 + 8}}{4}$$

$$x = \frac{3 \pm \sqrt{17}}{4}$$

5. $6x^2 - 2x + 1 = 0$
 $\downarrow \quad \downarrow \quad \nearrow$
 $a = 6 \quad b = -2 \quad c = 1$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(6)(1)}}{2(6)}$$

$$x = \frac{2 \pm \sqrt{4 - 24}}{12}$$

$$x = \frac{2 \pm \sqrt{-20}}{12}$$

Since $\sqrt{-20}$ is not a real number, this equation does not have a real number solution.

6. $-4x^2 - 7x - 3 = 0$
 $\downarrow \quad \downarrow \quad \nearrow$
 $a = -4 \quad b = -7 \quad c = -3$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(-4)(-3)}}{2(-4)}$$

$$x = \frac{7 \pm \sqrt{49 - 48}}{-8}$$

$$x = \frac{7 \pm \sqrt{1}}{-8}$$

$$x = \frac{7 \pm 1}{-8}$$

$$x = -\frac{7 \pm 1}{8}$$

$$x = -\frac{8}{8} \text{ or } x = -\frac{6}{8}$$

$$x = -1 \quad x = -\frac{3}{4}$$

Extensions

$$2\sqrt{3}x^2 - \sqrt{2}x - 3\sqrt{7} = 0$$

$$a = 2\sqrt{3} \quad b = -\sqrt{2} \quad c = -3\sqrt{7}$$

$$x = \frac{-(-\sqrt{2}) \pm \sqrt{(-\sqrt{2})^2 - 4(2\sqrt{3})(-3\sqrt{7})}}{2(2\sqrt{3})}$$

$$x = \frac{\sqrt{2} \pm \sqrt{2 + 24\sqrt{21}}}{4\sqrt{3}}$$

$$\text{The two roots are } \frac{\sqrt{2} + \sqrt{2 + 24\sqrt{21}}}{4\sqrt{3}} \text{ and } \frac{\sqrt{2} - \sqrt{2 + 24\sqrt{21}}}{4\sqrt{3}}.$$

The approximate value of these roots is as follows:

$$\frac{\sqrt{2} + \sqrt{2 + 24\sqrt{21}}}{4\sqrt{3}} = \frac{1.414\,213\,562 + \sqrt{2 + 24(4.582\,575\,695)}}{4(1.732\,050\,808)}$$

$$= \frac{1.414\,213\,562 + \sqrt{2 + 109.981\,816\,7}}{6.928\,203\,23}$$

$$= \frac{1.414\,213\,562 + \sqrt{111.981\,816\,7}}{6.928\,203\,23}$$

$$= \frac{1.414\,213\,562 + 10.582\,146\,13}{6.928\,203\,23}$$

$$= \frac{11.996\,359\,69}{6.928\,203\,23}$$

$$= 1.731\,525\,374$$

$$\approx 1.73$$

$$\frac{\sqrt{2} - \sqrt{2 + 24\sqrt{21}}}{4\sqrt{3}} = \frac{1.414\,213\,562 - \sqrt{2 + 24(4.582\,575\,695)}}{4(1.732\,050\,808)}$$

$$= \frac{1.414\,213\,562 - \sqrt{2 + 109.981\,816\,7}}{6.928\,203\,23}$$

$$= \frac{1.414\,213\,562 - \sqrt{111.981\,816\,7}}{6.928\,203\,23}$$

$$= \frac{1.414\,213\,562 - 10.582\,146\,13}{6.928\,203\,23}$$

$$= \frac{-9.167\,932\,568}{6.928\,203\,23}$$

$$\approx -1.323\,277\,084$$

$$\approx -1.32$$

The two approximate roots rounded to two decimal places are 1.73 and -1.32.



Exploring Topic 6

Activity 1

Solve radical equations and equations involving rational expressions both of which transform into quadratic equations.

1. A radical equation is any equation containing a variable under a radical sign.

2. When squaring both sides of an equation, extraneous (extra) roots are sometimes introduced. All possible roots must be verified before they can be considered as a solution to the equation.

3. a. $x + \sqrt{x+3} = 3$

$$\sqrt{x+3} = 3 - x$$

$$(\sqrt{x+3})^2 = (3-x)^2$$

$$x+3 = 9 - 6x + x^2$$

$$0 = x^2 - 7x + 6$$

$$0 = (x-6)(x-1)$$

$$x-6=0 \text{ or } x-1=0$$

$$x=6 \quad x=1$$

Check:

$$x=6$$

LS	RS	LS	RS
$x + \sqrt{x+3}$	3	$x + \sqrt{x+3}$	3
$6 + \sqrt{6+3}$	3	$1 + \sqrt{1+3}$	3
$6 + \sqrt{9}$	3	$1 + \sqrt{4}$	3
$6 + 3$	3	$1 + 2$	3
9	3	3	3
LS $\neq$ RS		LS = RS	

The solution to this equation is $x=1$. The root $x=6$ is extraneous.

b. $\sqrt{2x+4} + 3 = x+5$

$$\sqrt{2x+4} = x+2$$

$$(\sqrt{2x+4})^2 = (x+2)^2$$

$$2x+4 = x^2 + 4x + 4$$

$$0 = x^2 + 2x$$

$$0 = x(x+2)$$

$$x=0 \text{ or } x+2=0$$

$$x=-2$$

Check:

$$x=0$$

$$x=-2$$

LS	RS	LS	RS
$\sqrt{2x+4} + 3$	$x+5$	$\sqrt{2x+4} + 3$	$x+5$
$\sqrt{0+4} + 3$	$0+5$	$\sqrt{2(-2)+4} + 3$	$-2+5$
$\sqrt{4} + 3$	5	$\sqrt{-4+4} + 3$	3
$2+3$	5	$\sqrt{0} + 3$	3
5	5	$0+3$	3
LS = RS		3	3
		LS = RS	

The solutions are $x=0$ and $x=-2$.

c. $4 - \sqrt{3x - 14} = 7 - x$

$$-\sqrt{3x - 14} = 3 - x$$

$$(-\sqrt{3x - 14})^2 = (3 - x)^2$$

$$3x - 14 = 9 - 6x + x^2$$

$$0 = x^2 - 9x + 23$$

$$a = 1, b = -9, c = 23$$

$$x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(23)}}{2(1)}$$

$$x = \frac{9 \pm \sqrt{81 - 92}}{2}$$

$$x = \frac{9 \pm \sqrt{-11}}{2}$$

This equation has no real roots because $\sqrt{-11}$ is not a real number.

d. $x - \sqrt{x} = 6$

$$-\sqrt{x} = 6 - x$$

$$(-\sqrt{x})^2 = (6 - x)^2$$

$$x = 36 - 12x + x^2$$

$$0 = x^2 - 13x + 36$$

$$0 = (x - 4)(x - 9)$$

$$x - 4 = 0 \text{ or } x - 9 = 0$$

$$x = 4 \quad x = 9$$

Check:

$$x = 4$$

LS	RS
$x - \sqrt{x}$	6
$4 - \sqrt{4}$	6
$4 - 2$	6
2	6
LS $\neq$ RS	

$$x = 9$$

LS	RS
$x - \sqrt{x}$	6
$9 - \sqrt{9}$	6
$9 - 3$	6
6	6
LS = RS	

The root of the equation is 9. The root $x = 4$ is an extraneous root.

e. $\sqrt{3z+6} - z = -4$

$$\sqrt{3z+6} = z - 4$$

$$(\sqrt{3z+6})^2 = (z-4)^2$$

$$3z+6 = z^2 - 8z + 16$$

$$0 = z^2 - 11z + 10$$

$$0 = (z-10)(z-1)$$

$$z-10=0 \text{ or } z-1=0$$

$$z=10 \quad z=1$$

Check:

$$z=10$$

$$z=1$$

LS	RS	LS	RS
$\sqrt{3z+6} - z$	-4	$\sqrt{3z+6} - z$	-4
$\sqrt{3(10)+6} - 10$	-4	$\sqrt{3(1)+6} - 1$	-4
$\sqrt{30+6} - 10$	-4	$\sqrt{3+6} - 1$	-4
$\sqrt{36} - 10$	-4	$\sqrt{9} - 1$	-4
$6 - 10$	-4	$3 - 1$	-4
-4	-4	2	-4
LS = RS	RS	LS $\neq$ RS	RS

The root is 10. The root $z=1$ is an extraneous root.

f. $x - 2\sqrt{x-5} = 5$

$$-2\sqrt{x-5} = 5 - x$$

$$(-2\sqrt{x-5})^2 = (5-x)^2$$

$$4(x-5) = 25 - 10x + x^2$$

$$4x - 20 = 25 - 10x + x^2$$

$$0 = x^2 - 14x + 45$$

$$0 = (x-9)(x-5)$$

$$x-9=0 \text{ or } x-5=0$$

$$x=9 \quad x=5$$

Check:

$$x=9$$

$$x=5$$

LS	RS	LS	RS
$x - 2\sqrt{x-5}$	5	$x - 2\sqrt{x-5}$	5
$9 - 2\sqrt{9-5}$	5	$5 - 2\sqrt{5-5}$	5
$9 - 2\sqrt{4}$	5	$5 - 2\sqrt{0}$	5
$9 - 2(2)$	5	$5 - 2(0)$	5
$9 - 4$	5	$5 - 0$	5
5	5	5	5
LS = RS	RS	LS = RS	RS

The roots are 9 and 5.

g. $a + \sqrt{a+9} = 2a-3$

$$\sqrt{a+9} = a-3$$

$$(\sqrt{a+9})^2 = (a-3)^2$$

$$a+9 = a^2 - 6a + 9$$

$$0 = a^2 - 7a$$

$$0 = a(a-7)$$

$$a = 0 \text{ or } a - 7 = 0$$

$$a = 7$$

Check:

$$a = 0$$

$$a = 7$$

LS	RS	LS	RS
$a + \sqrt{a+9}$	$2a-3$	$a + \sqrt{a+9}$	$2a-3$
$0 + \sqrt{0+9}$	$2(0)-3$	$7 + \sqrt{7+9}$	$2(7)-3$
$0 + \sqrt{9}$	$0-3$	$7 + \sqrt{16}$	$14-3$
3	-3	7+4	11
LS $\neq$ RS	RS	11	11
		LS = RS	RS

The root is 7. The root $a = 0$ is extraneous.

h.

$$x+3 = \sqrt{2x+19}$$

$$(x+3)^2 = (\sqrt{2x+19})^2$$

$$x^2 + 6x + 9 = 2x + 19$$

$$x^2 + 4x - 10 = 0$$

$$a = 1, b = 4, c = -10$$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-10)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16+40}}{2}$$

$$x = \frac{-4 \pm \sqrt{56}}{2}$$

$$x = \frac{-4 \pm 2\sqrt{14}}{2}$$

$$x = \frac{-2 \pm \sqrt{14}}{1}$$

$$x = -2 \pm \sqrt{14}$$

Check:

$$x = -2 + \sqrt{14}$$

$$x \doteq -2 + 3.741\,657\,387$$

$$x \doteq 1.741\,657\,387$$

LS	RS
$x + 3$	$\sqrt{2x + 19}$
$1.741\,657\,387 + 3$	$\sqrt{2(1.741\,657\,387) + 19}$
$4.741\,657\,387$	$\sqrt{22.483\,314\,77}$
$4.741\,657\,387$	$4.741\,657\,387$
LS	RS

$$x = -2 - \sqrt{14}$$

$$x \doteq -2 - 3.741\,657\,387$$

$$x \doteq -5.741\,657\,387$$

LS	RS
$x + 3$	$\sqrt{2x + 19}$
$-5.741\,657\,387 + 3$	$\sqrt{2(-5.741\,657\,387) + 19}$
$-2.741\,657\,387$	$\sqrt{7.516\,685\,227}$
$-2.741\,657\,387$	$2.741\,657\,387$
LS	RS

The root is $-2 + \sqrt{14}$. The root $x = -2 - \sqrt{14}$ is extraneous.

4. You should first identify the nonpermissible values if there are any.

5. Let the denominator equal zero and solve for the variable.

$$6. \quad \frac{4}{x} + \frac{x}{4} = \frac{5}{2} \quad (x \neq 0)$$

$$\text{L.C.D.} = 4x$$

$$4x \left(\frac{4}{x} + \frac{x}{4} \right) = 4x \left(\frac{5}{2} \right)$$

$$4x \left(\frac{4}{x} + \frac{x}{4} \right) = 4x \left(\frac{5}{2} \right)$$

$$4(4) + x(x) = 2x(5)$$

$$16 + x^2 = 10x$$

$$x^2 - 10x + 16 = 0$$

$$(x - 8)(x - 2) = 0$$

$$x - 8 = 0 \quad \text{or} \quad x - 2 = 0$$

$$x = 8 \quad x = 2$$

The roots are 8 and 2.

$$\text{b. } \frac{3}{x+2} = \frac{1}{4} - \frac{1}{x-4} \quad (x \neq -2, 4)$$

$$\text{L.C.D.} = 4(x+2)(x-4)$$

$$4(x+2)(x-4)\left(\frac{3}{x+2}\right) = 4(x+2)(x-4)\left(\frac{1}{4} - \frac{1}{x-4}\right)$$

$$4\left(\frac{1}{x+2}\right)(x-4)\left(\frac{3}{x+2}\right) = 4\left(\frac{1}{x+2}\right)(x-4)\left(\frac{1}{4} - \frac{1}{x-4}\right)$$

$$-4(x+2)\left(\frac{1}{x+2}\right)\left(\frac{1}{x-4}\right)$$

$$4(x-4)(3) = (x+2)(x-4) - 4(x+2)$$

$$12x - 48 = x^2 - 2x - 8 - 4x - 8$$

$$12x - 48 = x^2 - 6x - 16$$

$$0 = x^2 - 18x + 32$$

$$0 = (x-16)(x-2)$$

$$x-16=0 \text{ or } x-2=0$$

$$x=16 \quad x=2$$

The roots are 16 and 2.

$$\text{c. } \frac{5x+7}{3x+2} = \frac{x-1}{x-2} \quad \left(x \neq -\frac{2}{3}, 2\right)$$

$$\text{L.C.D.} = (3x+2)(x-2)$$

$$\left(\frac{1}{3x+2}\right)(x-2)\left(\frac{5x+7}{3x+2}\right) = (3x+2)\left(\frac{1}{x-2}\right)\left(\frac{x-1}{x-2}\right)$$

$$(x-2)(5x+7) = (3x+2)(x-1)$$

$$5x^2 - 3x - 14 = 3x^2 - x - 2$$

$$2x^2 - 2x - 12 = 0$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0$$

$$x-3=0 \text{ or } x+2=0$$

$$x=3 \quad x=-2$$

The roots are 3 and -2.

d. $\frac{1-5x}{-3x} = \frac{x+1}{2}$

$(x \neq 0)$

L.C.D. = $-6x$

$$\frac{2}{-6x} \left(\frac{1-5x}{-3x} \right) = \frac{-3}{-6x} \left(\frac{x+1}{2} \right)$$

$$2(1-5x) = -3x(x+1)$$

$$2 - 10x = -3x^2 - 3x$$

$$3x^2 - 7x + 2 = 0$$

$$(3x-1)(x-2) = 0$$

$$3x-1=0 \text{ or } x-2=0$$

$$3x=1 \quad x=2$$

$$x = \frac{1}{3}$$

The roots are $\frac{1}{3}$ and 2.

e. $\frac{6}{x+1} - \frac{5}{1-x} = 7$

$(x \neq -1, 1)$

L.C.D. = $(x+1)(1-x)$

$$(x+1)(1-x) \left(\frac{6}{x+1} - \frac{5}{1-x} \right) = (x+1)(1-x)(7)$$

$$\left(\frac{1}{x+1} \right) (1-x) \left(\frac{6}{-x+1} \right) - (x+1) \left(\frac{1}{1-x} \right) \left(\frac{5}{1-x} \right) = (x+1)(1-x)(7)$$

$$(1-x)(6) - (x+1)(5) = (x+1)(1-x)(7)$$

$$6 - 6x - 5x - 5 = (-x^2 + 1)(7)$$

$$-11x + 1 = -7x^2 + 7$$

$$7x^2 - 11x - 6 = 0$$

$$(7x+3)(x-2) = 0$$

$$7x+3=0 \text{ or } x-2=0$$

$$7x=-3 \quad x=2$$

$$x = -\frac{3}{7}$$

The roots are $-\frac{3}{7}$ and 2.

$$f. \frac{t+2}{t} - \frac{4t}{3} + 2 = 0$$

$$(t \neq 0)$$

$$\text{L.C.D.} = 3t$$

$$3t \left(\frac{t+2}{t} - \frac{4t}{3} + 2 \right) = 3t(0)$$

$$3t \left(\frac{t+2}{t} - \frac{4t}{3} + 2 \right) = 3t(0)$$

$$3(t+2) - t(4t) + 6t = 0$$

$$3t + 6 - 4t^2 + 6t = 0$$

$$-4t^2 + 9t + 6 = 0$$

$$4t^2 - 9t - 6 = 0$$

$$a = 4, b = -9, c = -6$$

$$t = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(4)(-6)}}{2(4)}$$

$$t = \frac{9 \pm \sqrt{81 + 96}}{8}$$

$$t = \frac{9 \pm \sqrt{177}}{8}$$

$$\text{The roots are } \frac{9 + \sqrt{177}}{8} \text{ and } \frac{9 - \sqrt{177}}{8}.$$

$$g. \frac{n+1}{2} - \frac{n-1}{3} = \frac{2n+3}{4n}$$

$$(n \neq 0)$$

$$\text{L.C.D.} = 12n$$

$$12n \left(\frac{n+1}{2} - \frac{n-1}{3} \right) = 12n \left(\frac{2n+3}{4n} \right)$$

$$12n \left(\frac{n+1}{2} - \frac{n-1}{3} \right) = 12n \left(\frac{2n+3}{4n} \right)$$

$$6n(n+1) - 4n(n-1) = 3(2n+3)$$

$$6n^2 + 6n - 4n^2 + 4n = 6n + 9$$

$$2n^2 + 4n - 9 = 0$$

$$a = 2, b = 4, c = -9$$

$$x = \frac{-4 \pm \sqrt{(+4)^2 - 4(2)(-9)}}{2(2)}$$

$$x = \frac{-4 \pm \sqrt{16 + 72}}{4}$$

$$x = \frac{-4 \pm \sqrt{88}}{4}$$

$$x = \frac{-4 \pm 2\sqrt{22}}{4}$$

$$x = \frac{-2 \pm \sqrt{22}}{2}$$

$$x = \frac{-2 \pm \sqrt{22}}{2}$$

$$\text{The roots are } \frac{-2 + \sqrt{22}}{2} \text{ and } \frac{-2 - \sqrt{22}}{2}.$$

$$h. \frac{1-x}{2x} = \frac{5x}{3x+1}$$

$$\left(x \neq 0, -\frac{1}{3}\right) \quad \text{L.C.D.} = 2x(3x+1)$$

$$2x(3x+1) \left(\frac{1-x}{2x} \right) = 2x(3x+1) \left(\frac{5x}{3x+1} \right)$$

$$(3x+1)(1-x) = 2x(5x)$$

$$-3x^2 + 2x + 1 = 10x^2$$

$$-13x^2 + 2x + 1 = 0$$

$$a = -13, b = 2, c = 1$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(-13)(1)}}{2(-13)}$$

$$x = \frac{-2 \pm \sqrt{4 + 52}}{-26}$$

$$x = \frac{-2 \pm \sqrt{56}}{-26}$$

$$x = \frac{2 \pm \sqrt{56}}{26}$$

$$x = \frac{2 \pm 2\sqrt{14}}{26}$$

$$x = \frac{\cancel{2}(1 \pm \sqrt{14})}{\cancel{26}_{13}}$$

$$x = \frac{1 \pm \sqrt{14}}{13}$$

The roots are $\frac{1+\sqrt{14}}{13}$ and $\frac{1-\sqrt{14}}{13}$.

$$i. \quad x - \frac{2x-1}{3} = x^2 - \frac{1}{2} \quad \text{L.C.D.} = 6$$

$$6\left(x - \frac{2x-1}{3}\right) = 6\left(x^2 - \frac{1}{2}\right)$$

$$6(x) - \cancel{6}^2 \left(\frac{2x-1}{\cancel{3}_1} \right) = 6(x^2) - \cancel{6}^3 \left(\frac{1}{\cancel{2}_1} \right)$$

$$6x - 2(2x-1) = 6x^2 - 3$$

$$6x - 4x + 2 = 6x^2 - 3$$

$$0 = 6x^2 - 2x - 5$$

$$a = 6, b = -2, c = -5$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(6)(-5)}}{2(6)}$$

$$x = \frac{2 \pm \sqrt{4 + 120}}{12}$$

$$x = \frac{2 \pm \sqrt{124}}{12}$$

$$x = \frac{2 \pm 2\sqrt{31}}{12}$$

$$x = \frac{\cancel{2}(1 \pm \sqrt{31})}{\cancel{12}_6}$$

$$x = \frac{1 \pm \sqrt{31}}{6}$$

The roots are $\frac{1+\sqrt{31}}{6}$ and $\frac{1-\sqrt{31}}{6}$.

j. $\frac{x+3}{2x-7} = \frac{2x-1}{x-3}$

$\left(x \neq \frac{7}{2}, 3\right)$

L.C.D. = $(2x-7)(x-3)$

$$\left(\frac{2x+7}{2x-7}\right)(x-3) \left(\frac{x+3}{2x-7}\right) = (2x-7) \left(\frac{1}{x-3}\right) \left(\frac{2x-1}{x-3}\right)$$

$$(x-3)(x+3) = (2x-7)(2x-1)$$

$$x^2 - 9 = 4x^2 - 16x + 7$$

$$0 = 3x^2 - 16x + 16$$

$$0 = (3x-4)(x-4)$$

$$3x-4=0 \text{ or } x-4=0$$

$$3x=4 \quad x=4$$

$$x = \frac{4}{3}$$

The roots are $\frac{4}{3}$ and 4.

7. a. Let the number be x .

Then, $x - 28\left(\frac{1}{x}\right) = 3$

$$x - \frac{28}{x} = 3 \quad (\text{L.C.D.} = x, x \neq 0)$$

$$x\left(x - \frac{28}{x}\right) = x(3)$$

$$x(x) - \cancel{x} \left(\frac{28}{\cancel{x}}\right) = x(3)$$

$$x^2 - 28 = 3x$$

$$x^2 - 3x - 28 = 0$$

$$(x-7)(x+4) = 0$$

$$x-7=0 \text{ or } x+4=0$$

$$x=7 \quad x=-4$$

The number is 7 or -4.

b. $d = \sqrt{(x-h)^2 + (y-k)^2} \quad h=3, y=2, k=-2, d=5$

$$5 = \sqrt{(x-3)^2 + (2-(-2))^2}$$

$$5 = \sqrt{(x-3)^2 + (4)^2}$$

$$(5)^2 = \left(\sqrt{(x-3)^2 + 16}\right)^2$$

$$25 = (x-3)^2 + 16$$

$$25 = x^2 - 6x + 9 + 16$$

$$25 = x^2 - 6x + 25$$

$$0 = x^2 - 6x$$

$$0 = x(x-6)$$

$$x = 0 \text{ or } x - 6 = 0$$

$$x = 6$$

Check:

$$x = 6$$

LS	RS
d	$\sqrt{(x-h)^2 + (y-k)^2}$
5	$\sqrt{(6-3)^2 + [2-(-2)]^2}$
5	$\sqrt{(3)^2 + (4)^2}$
5	$\sqrt{9+16}$
5	$\sqrt{25}$
5	5
LS = RS	

$$x = 0$$

LS	RS
d	$\sqrt{(x-h)^2 + (y-k)^2}$
5	$\sqrt{(0-3)^2 + [2-(-2)]^2}$
5	$\sqrt{(-3)^2 + 4^2}$
5	$\sqrt{9+16}$
5	$\sqrt{25}$
5	5
LS = RS	

The value of x is 6 or 0.

c. $y = \sqrt{2x^2 - 5x + 33}$

When $y = 6$,

$$6 = \sqrt{2x^2 - 5x + 33}$$

$$(6)^2 = (\sqrt{2x^2 - 5x + 33})^2$$

$$36 = 2x^2 - 5x + 33$$

$$0 = 2x^2 - 5x - 3$$

$$0 = (2x+1)(x-3)$$

$$2x + 1 = 0 \text{ or } x - 3 = 0$$

$$2x = -1 \quad x = 3$$

$$x = -\frac{1}{2}$$

Check:

$$x = -\frac{1}{2}$$

LS	RS
y	$\sqrt{2x^2 - 5x + 33}$
6	$\sqrt{2\left(-\frac{1}{2}\right)^2 - 5\left(-\frac{1}{2}\right) + 33}$
6	$\sqrt{2\left(\frac{1}{4}\right) + \frac{5}{2} + 33}$
6	$\sqrt{\frac{1}{2} + 2\frac{1}{2} + 33}$
6	$\sqrt{36}$
6	6
LS = RS	

$$x = 3$$

LS	RS
y	$\sqrt{2x^2 - 5x + 33}$
6	$\sqrt{2(3)^2 - 5(3) + 33}$
6	$\sqrt{2(9) - 15 + 33}$
6	$\sqrt{18 - 15 + 33}$
6	$\sqrt{36}$
6	6
LS = RS	

The value of x is $-\frac{1}{2}$ or 3.

d. Let x be the mark that the student needs.

$$\text{Then, } \frac{72 + 56 + 86 + x}{4} = 75 \quad (\text{L. C. D.} = 4)$$

$$\frac{1}{4} \left(\frac{72 + 56 + 86 + x}{4} \right) = 4(75)$$

$$214 + x = 300$$

$$x = 300 - 214$$

$$x = 86$$

The student needs 86% on the last exam to have an average of 75% on all four exams.

e. $\sqrt{\sqrt{x^2 - 3x} - 3x} = \sqrt{2x}$

$(\sqrt{\sqrt{x^2 - 3x}})^2 = (\sqrt{2x})^2$
(Square both sides.)

$\sqrt{x^2 - 3x} = 2x$

$(\sqrt{x^2 - 3x})^2 = (2x)^2$
(Square both sides again.)

$x^2 - 3x = 4x^2$

$0 = 3x^2 + 3x$

$0 = 3x(x+1)$

$3x = 0$ or $x+1 = 0$

$x = 0$ $x = -1$

Check:

$x = 0$

LS	RS
$\sqrt{\sqrt{x^2 - 3x}}$	$\sqrt{2x}$
$\sqrt{\sqrt{0^2 - 3(0)}}$	$\sqrt{2(0)}$
$\sqrt{\sqrt{0}}$	$\sqrt{0}$
$\sqrt{0}$	0
0	0
LS	RS

$x = -1$

LS	RS
$\sqrt{\sqrt{x^2 - 3x}}$	$\sqrt{2x}$
$\sqrt{\sqrt{(-1)^2 - 3(-1)}}$	$\sqrt{2(-1)}$
$\sqrt{\sqrt{1+3}}$	$\sqrt{-2}$
$\sqrt{\sqrt{4}}$	$\sqrt{-2}$
$\sqrt{2}$	$\sqrt{-2}$
LS	RS

Since $\sqrt{-2}$ is not a real number, $x = -1$ is not a real number solution.

The solution is $x = 0$.

Extra Help

1. a. $\sqrt{\frac{x}{6}} + 4 = 8$
 $\left(\sqrt{\frac{x}{6}}\right)^2 = (4)^2$
 $\frac{x}{6} = 16$
 $x = 96$

Check:

$$x = 96$$

LS	RS
$\sqrt{\frac{x}{6}} + 4$	8
$\sqrt{\frac{96}{6}} + 4$	8
$\sqrt{16} + 4$	8
$4 + 4$	8
8	8
LS	RS

The solution is 96.

Check:

$$x = \frac{4}{27}$$

LS	RS
$\sqrt{3x}$	$\frac{2}{3}$
$\sqrt[3]{\frac{4}{27}}$	$\frac{2}{3}$
$\sqrt{\frac{4}{9}}$	$\frac{2}{3}$
$\frac{2}{3}$	$\frac{2}{3}$
LS	RS

The solution is $\frac{4}{27}$.

b. $\sqrt{3x} = \frac{2}{3}$

$$(\sqrt{3x})^2 = \left(\frac{2}{3}\right)^2$$

$$3x = \frac{4}{9}$$

$$\frac{3x}{3} = \frac{4}{3(9)}$$

$$x = \frac{4}{27}$$

c. $\sqrt{5x+3} + 7 = 9$

$$(\sqrt{5x+3})^2 = (2)^2$$

$$5x+3 = 4$$

$$5x = 1$$

$$x = \frac{1}{5}$$

Check:

$$x = \frac{1}{5}$$

LS	RS
$\sqrt{5x+3} + 7$	9
$\sqrt{5 \times \frac{1}{5} + 3} + 7$	9
$\sqrt{1+3} + 7$	9
$\sqrt{4} + 7$	9
$2+7$	9
9	9
LS	RS

The solution is $\frac{1}{5}$.

d. $\sqrt{7x-9} = \sqrt{2x+21}$

$$(\sqrt{7x-9})^2 = (\sqrt{2x+21})^2$$

$$7x-9 = 2x+21$$

$$5x-9 = 21$$

$$5x = 30$$

$$x = 6$$

Check:

$$x = 6$$

LS	RS
$\sqrt{7x-9}$	$\sqrt{2x+21}$
$\sqrt{7(6)-9}$	$\sqrt{2(6)+21}$
$\sqrt{42-9}$	$\sqrt{12+21}$
$\sqrt{33}$	$\sqrt{33}$
LS	RS

The solution is 6.

e. $\sqrt{x^2+3x-2} = 0$

$$\sqrt{x^2+3x} = 2$$

$$(\sqrt{x^2+3x})^2 = (2)^2$$

$$x^2+3x = 4$$

$$x^2+3x-4 = 0$$

$$(x+4)(x-1) = 0$$

$$x = -4 \text{ or } x = 1$$

Check:

$$x = -4$$

LS	RS
$\sqrt{x^2 + 3x} - 2$	0
$\sqrt{(-4)^2 + 3(-4)} - 2$	0
$\sqrt{16 - 12} - 2$	0
$\sqrt{4} - 2$	0
$2 - 2$	0
0	0
LS = RS	

f. $\sqrt{3x+10} - x = 0$

$$\sqrt{3x+10} = x$$

$$(\sqrt{3x+10})^2 = (x)^2$$

$$3x+10 = x^2$$

$$0 = x^2 - 3x - 10$$

$$0 = (x-5)(x+2)$$

$$x = 5 \text{ or } x = -2$$

Check:

$$x = 5$$

LS	RS
$\sqrt{3x+10} - x$	0
$\sqrt{3(5)+10} - 5$	0
$\sqrt{15+10} - 5$	0
$\sqrt{25} - 5$	0
$5 - 5$	0
0	0
LS = RS	

$$x = 1$$

LS	RS
$\sqrt{x^2 + 3x} - 2$	0
$\sqrt{(1)^2 + 3(1)} - 2$	0
$\sqrt{1+3} - 2$	0
$\sqrt{4} - 2$	0
$2 - 2$	0
0	0
LS = RS	

The solutions are -4 and 1.

$$x = -2$$

LS	RS
$\sqrt{3x+10} - x$	0
$\sqrt{3(-2)+10} - (-2)$	0
$\sqrt{-6+10} + 2$	0
$\sqrt{4} + 2$	0
$2 + 2$	0
4	0
LS	$\neq$ RS

The solution is 5. The solution $x = -2$ is extraneous.

g. $\sqrt{x-3} + 3 = x$

$$(\sqrt{x-3})^2 = (x-3)^2$$

$$x-3 = x^2 - 6x + 9$$

$$0 = x^2 - 7x + 12$$

$$0 = (x-3)(x-4)$$

$$x = 3 \text{ or } x = 4$$

Check:

$$x = 3$$

LS	RS
$\sqrt{x-3} + 3$	x
$\sqrt{3-3} + 3$	3
$\sqrt{0} + 3$	3
$0 + 3$	3
3	3
LS	= RS

$$x = 4$$

LS	RS
$\sqrt{x-3} + 3$	x
$\sqrt{4-3} + 3$	4
$\sqrt{1} + 3$	4
$1 + 3$	4
4	4
LS	= RS

The solutions are 3 and 4.

h. $(x+2)^2 = (\sqrt{18-x})^2$

$$x^2 + 4x + 4 = 18 - x$$

$$x^2 + 5x - 14 = 0$$

$$(x+7)(x-2) = 0$$

$$x = -7 \text{ or } x = 2$$

Check:

$$x = -7$$

LS	RS
$x+2$	$\sqrt{18-x}$
$-7+2$	$\sqrt{18-(-7)}$
-5	$\sqrt{25}$
-5	5
LS	RS

$$x = 2$$

LS	RS
$x+2$	$\sqrt{18-x}$
$2+2$	$\sqrt{18-2}$
4	$\sqrt{16}$
4	4
LS	RS

The solution is 2. The root $x = -7$ is extraneous.

i. $\sqrt{3x-5} - x = -5$

$$\sqrt{3x-5} = -5 + x$$

$$(\sqrt{3x-5})^2 = (-5+x)^2$$

$$3x-5 = 25-10x+x^2$$

$$0 = x^2 - 13x + 30$$

$$0 = (x-3)(x-10)$$

$$x = 3 \text{ or } x = 10$$

Check:

$$x = 3$$

$$x = 10$$

LS	RS	LS	RS
$\sqrt{3x-5} - x$	-5	$\sqrt{3x-5} - x$	-5
$\sqrt{3(3)-5} - 3$	-5	$\sqrt{3(10)-5} - 10$	-5
$\sqrt{9-5} - 3$	-5	$\sqrt{30-5} - 10$	-5
$\sqrt{4} - 3$	-5	$\sqrt{25} - 10$	-5
$2-3$	-5	$5-10$	-5
-1	-5	-5	-5
LS	RS	LS	RS

The solution is 10. The root $x = 3$ is extraneous.

j. $\sqrt{7x+25} = 1+x$

$$(\sqrt{7x+25})^2 = (1+x)^2$$

$$7x+25 = 1+2x+x^2$$

$$0 = x^2 - 5x - 24$$

$$0 = (x-8)(x+3)$$

$$x = 8 \text{ or } x = -3$$

Check:

$$x = 8$$

LS	RS
$\sqrt{7x+25}$	$1+x$
$\sqrt{7(8)+25}$	$1+8$
$\sqrt{56+25}$	9
$\sqrt{81}$	9
9	9
LS	RS

$$=$$

$$x = -3$$

LS	RS
$\sqrt{7x+25}$	$1+x$
$\sqrt{7(-3)+25}$	$1+(-3)$
$\sqrt{-21+25}$	-2
$\sqrt{4}$	-2
2	-2
LS	RS

$$\neq$$

The solution is 8. The root $x = 8$ is extraneous.

k. $\sqrt{2x-7} = 4$

$$(\sqrt{2x-7})^2 = (4)^2$$

$$2x-7 = 16$$

$$2x = 23$$

$$x = \frac{23}{2}$$

Check:

$$x = \frac{23}{2}$$

LS	RS
$\sqrt{2x-7}$	4
$\sqrt{2\left(\frac{23}{2}\right)-7}$	4
$\sqrt{23-7}$	4
$\sqrt{16}$	4
4	4
LS	RS

The solution is $\frac{23}{2}$.

- $$3\sqrt{x} = x - 4$$

$$(3\sqrt{x})^2 = (x - 4)^2$$

$$9x = x^2 - 8x + 16$$

$$0 = x^2 - 17x + 16$$

$$0 = (x - 16)(x - 1)$$

$$x = 16 \text{ or } x = 1$$

Check:

$$x = 16$$

LS	RS
$3\sqrt{x}$	$x - 4$
$3\sqrt{16}$	$16 - 4$
$3(4)$	12
12	12
LS	RS

The solution is 16. The root $x = 1$ is extraneous.

- $$a. \quad \frac{x}{x+4} = \frac{3}{x-1} \quad (x \neq -4, 1)$$

$$\text{L.C.D.} = (x+4)(x-1)$$

$$\left(\frac{1}{x+4}\right)(x-1)\left(\frac{x}{x+4}\right) = (x+4)\left(\frac{1}{x-1}\right)\left(\frac{3}{x-1}\right)$$

$$(x-1)(x) = (x+4)(3)$$

$$x^2 - x = 3x + 12$$

$$x^2 - 4x - 12 = 0$$

$$(x-6)(x+2) = 0$$

$$x = 6 \text{ or } x = -2$$

b. $\frac{6}{x+2} + \frac{1}{x-2} = 1 \quad (x \neq -2, 2)$

L.C.D. = $(x+2)(x-2)$

$$\left(\frac{1}{x+2}\right)(x-2)\left(\frac{6}{x+2}\right) + (x+2)\left(\frac{1}{x-2}\right)\left(\frac{1}{x-2}\right) = (x+2)(x-2)(1)$$

$$(x-2)(6) + (x+2)(1) = (x+2)(x-2)$$

$$6x - 12 + x + 2 = x^2 - 4$$

$$7x - 10 = x^2 - 4$$

$$0 = x^2 - 7x + 6$$

$$0 = (x-6)(x-1)$$

$$x = 6 \text{ or } x = 1$$

c. $\frac{3}{x} + \frac{2}{x-1} = 2 \quad (x \neq 0, 1)$

L.C.D. = $(x)(x-1)$

$$\left(\frac{1}{x}\right)(x-1)\left(\frac{3}{x}\right) + (x)\left(\frac{1}{x-1}\right)\left(\frac{2}{x-1}\right) = (x)(x-1)(2)$$

$$(x-1)(3) + (x)(2) = (x)(x-1)(2)$$

$$3x - 3 + 2x = 2x^2 - 2x$$

$$0 = 2x^2 - 7x + 3$$

$$0 = (2x-1)(x-3)$$

$$x = \frac{1}{2} \text{ or } x = 3$$

d. $2 + \frac{3}{x-5} = \frac{x}{x+3} \quad (x \neq 5, -3)$

L.C.D. = $(x-5)(x+3)$

$$(x-5)(x+3)(2) + \left(\frac{1}{x-5}\right)(x+3)\left(\frac{3}{x-5}\right) = (x-5)\left(\frac{1}{x+3}\right)\left(\frac{x}{x+3}\right)$$

$$2x^2 - 4x - 30 + 3x + 9 = x^2 - 5x$$

$$2x^2 - x - 21 = x^2 - 5x$$

$$x^2 + 4x - 21 = 0$$

$$(x+7)(x-3) = 0$$

$$x = -7 \text{ or } x = 3$$

$$e. \frac{x-1}{x+1} - \frac{6}{x-3} = 3$$

$$(x \neq -1, 3)$$

$$\text{L.C.D.} = (x+1)(x-3)$$

$$\left(\frac{1}{x+1} \right) (x-3) \left(\frac{x-1}{x+1} \right) - (x+1) \left(\frac{1}{x-3} \right) \left(\frac{6}{x-3} \right) = (x+1)(x-3)(3)$$

$$(x-3)(x-1) - (x+1)(6) = (x+1)(x-3)(3)$$

$$x^2 - 4x + 3 - 6x - 6 = 3x^2 - 6x - 9$$

$$x^2 - 10x - 3 = 3x^2 - 6x - 9$$

$$0 = 2x^2 + 4x - 6$$

$$0 = x^2 + 2x - 3$$

$$0 = (x+3)(x-1)$$

$$x = -3 \text{ or } x = 1$$

$$f. \frac{2}{x+3} + \frac{3}{x+4} = \frac{7}{(x+3)(x+4)}$$

$$(x \neq -3, -4)$$

$$\text{L.C.D.} = (x+3)(x+4)$$

$$\left(\frac{1}{x+3} \right) (x+4) \left(\frac{2}{x+3} \right) + (x+3) \left(\frac{1}{x+4} \right) \left(\frac{3}{x+4} \right) = \left(\frac{1}{x+3} \right) \left(\frac{1}{x+4} \right) \left(\frac{7}{(x+3)(x+4)} \right)$$

$$(x+4)(2) + (x+3)(3) = 7$$

$$2x + 8 + 3x + 9 = 7$$

$$5x + 17 = 7$$

$$5x = -10$$

$$x = -2$$

$$g. \quad x + \frac{4}{x} = 8 \quad (x \neq 0)$$

$$\text{L.C.D.} = x$$

$$x(x) + x \left(\frac{4}{x} \right) = x(8)$$

$$x^2 + 4 = 8x$$

$$x^2 - 8x + 4 = 0$$

$$a = 1, b = -8, c = 4$$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{8 \pm \sqrt{64 - 16}}{2}$$

$$x = \frac{8 \pm \sqrt{48}}{2}$$

$$x = \frac{8 \pm 4\sqrt{3}}{2}$$

$$x = \frac{2(4 \pm 2\sqrt{3})}{2}$$

$$x = 4 \pm 2\sqrt{3}$$

$$h. \quad \frac{3x-2}{x+5} = \frac{x-5}{3x+2} \quad \left(x \neq -5, -\frac{2}{3} \right)$$

$$\text{L.C.D.} = (x+5)(3x+2)$$

$$\left(\frac{1}{x+5} \right) (3x+2) \left(\frac{3x-2}{x+5} \right) = (x+5) \left(\frac{1}{3x+2} \right) \left(\frac{x-5}{3x+2} \right)$$

$$(3x+2)(3x-2) = (x+5)(x-5)$$

$$9x^2 - 4 = x^2 - 25$$

$$8x^2 - 4 = -25$$

$$8x^2 = -21$$

$$x^2 = \frac{-21}{8}$$

$$x = \pm \sqrt{\frac{-21}{8}}$$

Since $\sqrt{\frac{-21}{8}}$ is not a real number, there is no real number solution to this equation.

i. $\frac{4}{x-5} + \frac{1}{x+2} = \frac{2x+7}{(x-5)(x+2)}$

$(x \neq 5, -2)$

L.C.D. = $(x-5)(x+2)$

$$\left(\frac{1}{x-5}\right)(x+2)\left(\frac{4}{x-5}\right) + (x-5)\left(\frac{1}{x+2}\right)\left(\frac{1}{x+2}\right) = \left(\frac{1}{x-5}\right)\left(\frac{1}{x+2}\right)\left(\frac{2x+7}{(x-5)(x+2)}\right)$$

$$(x+2)(4) + (x-5)(1) = 2x+7$$

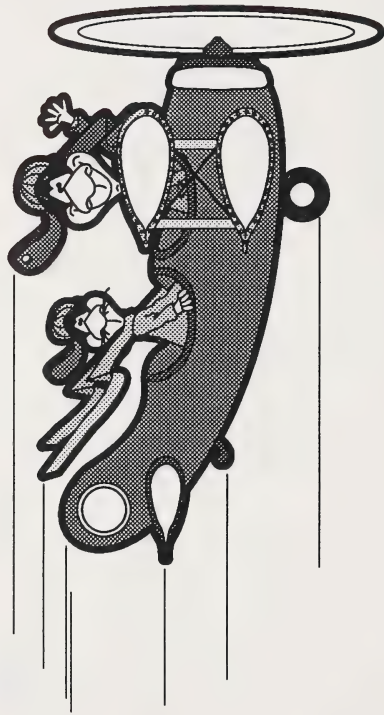
$$4x+8+x-5=2x+7$$

$$5x+3=2x+7$$

$$3x+3=7$$

$$3x=4$$

$$x=\frac{4}{3}$$



Extensions

$$1. \quad \sqrt{\sqrt{x^2 - 22}} = \sqrt{3\sqrt{x}}$$

$$\left(\sqrt{\sqrt{x^2 - 22}}\right)^2 = \left(\sqrt{3\sqrt{x}}\right)^2$$

$$\sqrt{x^2 - 22} = 3\sqrt{x}$$

$$\left(\sqrt{x^2 - 22}\right)^2 = \left(3\sqrt{x}\right)^2$$

$$x^2 - 22 = 9x$$

$$x^2 - 9x - 22 = 0$$

$$(x - 11)(x + 2) = 0$$

$$x = 11 \text{ or } x = -2$$

Check:

$$x = 11$$

LS	RS
$\sqrt{\sqrt{x^2 - 22}}$	$\sqrt{3\sqrt{x}}$
$\sqrt{\sqrt{(11)^2 - 22}}$	$\sqrt{3\sqrt{11}}$
$\sqrt{\sqrt{121 - 22}}$	$\sqrt{\sqrt{9 \times 11}}$
$\sqrt{\sqrt{99}}$	$\sqrt{\sqrt{99}}$
LS	RS

$$x = -2$$

LS	RS
$\sqrt{\sqrt{x^2 - 22}}$	$\sqrt{3\sqrt{x}}$
$\sqrt{\sqrt{(-2)^2 - 22}}$	$\sqrt{3\sqrt{-2}}$
$\sqrt{\sqrt{4 - 22}}$	$\sqrt{\sqrt{9 \times -2}}$
$\sqrt{\sqrt{-18}}$	$\sqrt{\sqrt{-18}}$

Since $\sqrt{\sqrt{-18}}$ is not a real number, $x = -2$ does not verify.

The solution is 11. The root $x = -2$ is extraneous.

2. $\sqrt{3x} - \sqrt{x-3} = 3$

$$-\sqrt{x-3} = 3 - \sqrt{3x}$$

$$(-\sqrt{x-3})^2 = (3 - \sqrt{3x})^2$$

$$x-3 = 9 - 6\sqrt{3x} + 3x$$

$$-2x - 12 = -6\sqrt{3x}$$

$$x+6 = 3\sqrt{3x}$$

$$(x+6)^2 = (3\sqrt{3x})^2$$

$$x^2 + 12x + 36 = 9(3x)$$

$$x^2 + 12x + 36 = 27x$$

$$x^2 - 15x + 36 = 0$$

$$(x-12)(x-3) = 0$$

$$x=12 \text{ or } x=3$$

Check:

$x=12$	LS	RS
	$\sqrt{3x} - \sqrt{x-3}$	3
	$\sqrt{3 \times 12} - \sqrt{12-3}$	3
	$\sqrt{36} - \sqrt{9}$	3
	$6-3$	3
	3	3
	LS =	RS

The solutions are 3 and 12.

3. $\sqrt{x+4} + \sqrt{x-4} = 4$

$$\sqrt{x+4} = 4 - \sqrt{x-4}$$

$$(\sqrt{x+4})^2 = (4 - \sqrt{x-4})^2$$

$$x+4 = 16 - 8\sqrt{x-4} + x-4$$

$$x+4 = 12 - 8\sqrt{x-4} + x$$

$$-8 = -8\sqrt{x-4}$$

$$1 = \sqrt{x-4}$$

$$(1)^2 = (\sqrt{x-4})^2$$

$$1 = x-4$$

$$5 = x$$

Check:

$x=5$	LS	RS
	$\sqrt{x+4} + \sqrt{x-4}$	4
	$\sqrt{5+4} + \sqrt{5-4}$	4
	$\sqrt{9} + \sqrt{1}$	4
	3+1	4
	4	4
	LS =	RS

The solution is 5.

4. $\sqrt{2x-3}-\sqrt{x-2}=1$

$$\sqrt{2x-3}=1+\sqrt{x-2}$$

$$(\sqrt{2x-3})^2=(1+\sqrt{x-2})^2$$

$$2x-3=1+2\sqrt{x-2}+x-2$$

$$2x-3=2\sqrt{x-2}+x-1$$

$$x-2=2\sqrt{x-2}$$

$$(x-2)^2=(2\sqrt{x-2})^2$$

$$x^2-4x+4=4(x-2)$$

$$x^2-4x+4=4x-8$$

$$x^2-8x+12=0$$

$$(x-6)(x-2)=0$$

$$x=6 \text{ or } x=2$$

Check:

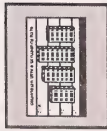
$$x=6$$

LS	RS
$\sqrt{2x-3}-\sqrt{x-2}$	1
$\sqrt{2 \times 6-3}-\sqrt{6-2}$	1
$\sqrt{12-3}-\sqrt{4}$	1
$\sqrt{9}-\sqrt{4}$	1
3-2	1
1	1
LS	RS

$$x=2$$

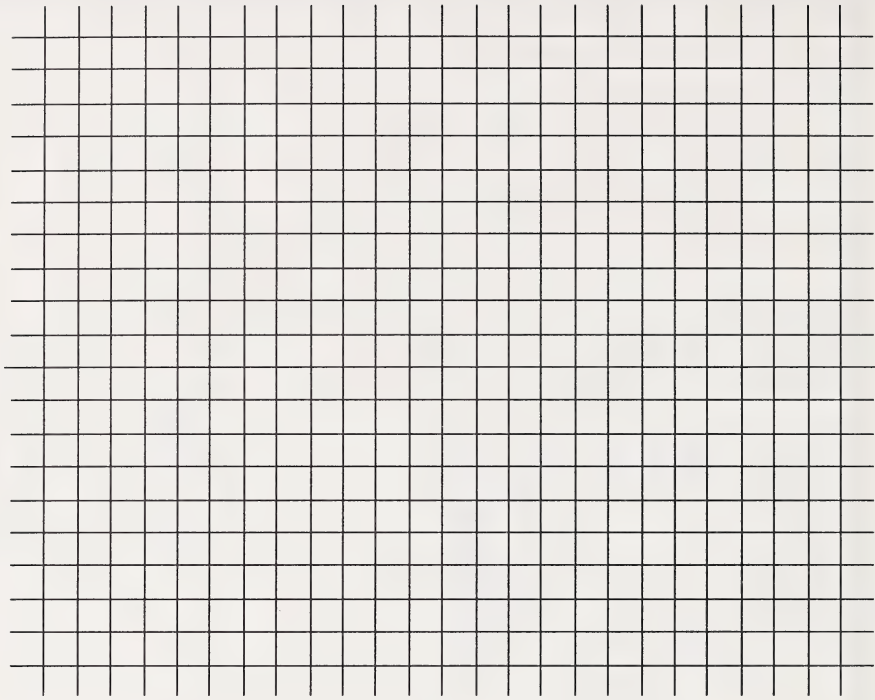
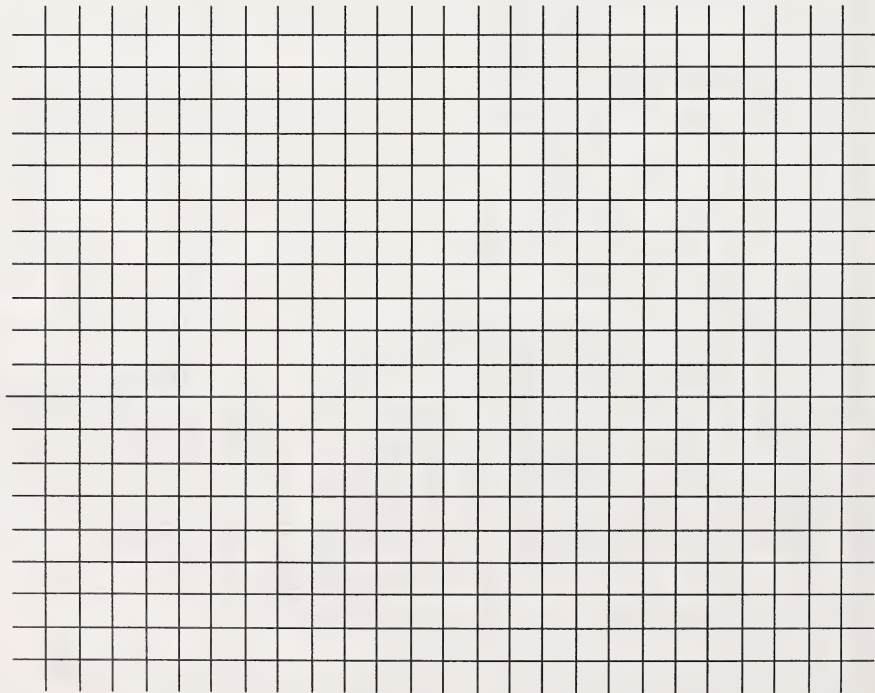
LS	RS
$\sqrt{2x-3}-\sqrt{x-2}$	1
$\sqrt{2 \times 2-3}-\sqrt{2-2}$	1
$\sqrt{4-3}-\sqrt{0}$	1
$\sqrt{1}-0$	1
1-0	1
1	1
LS	RS

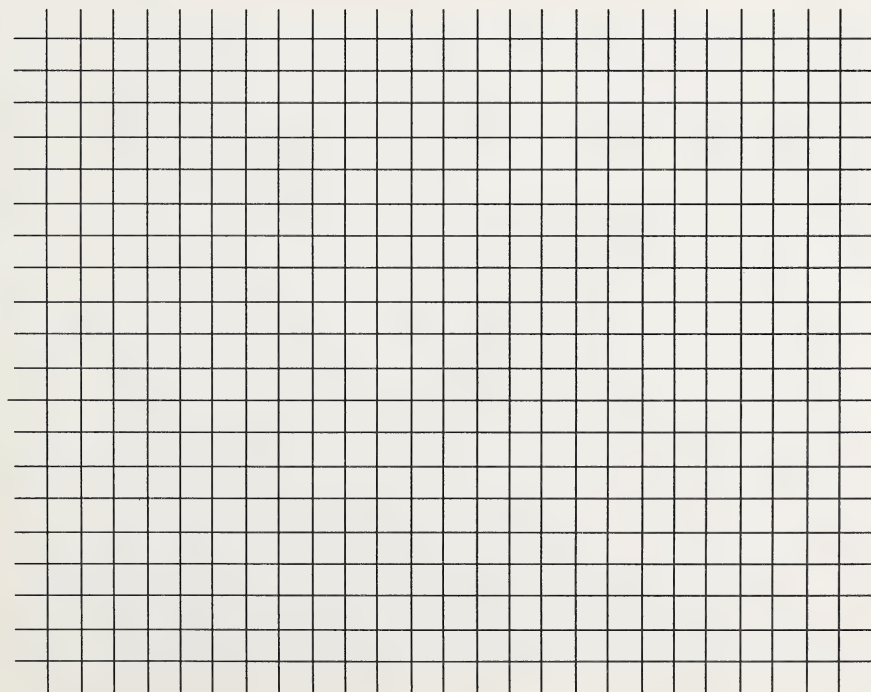
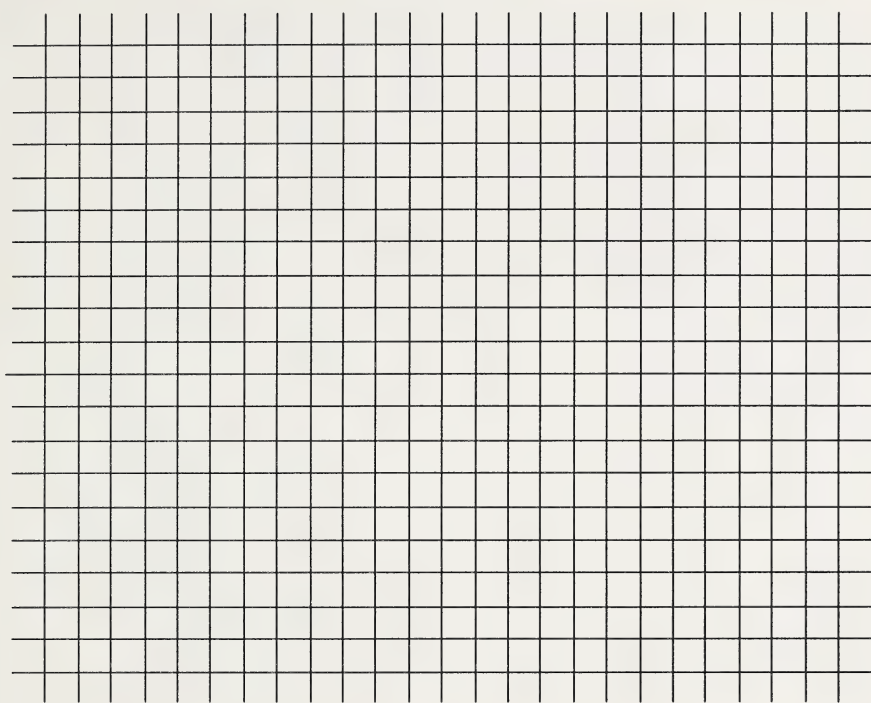
The solutions are 2 and 6.

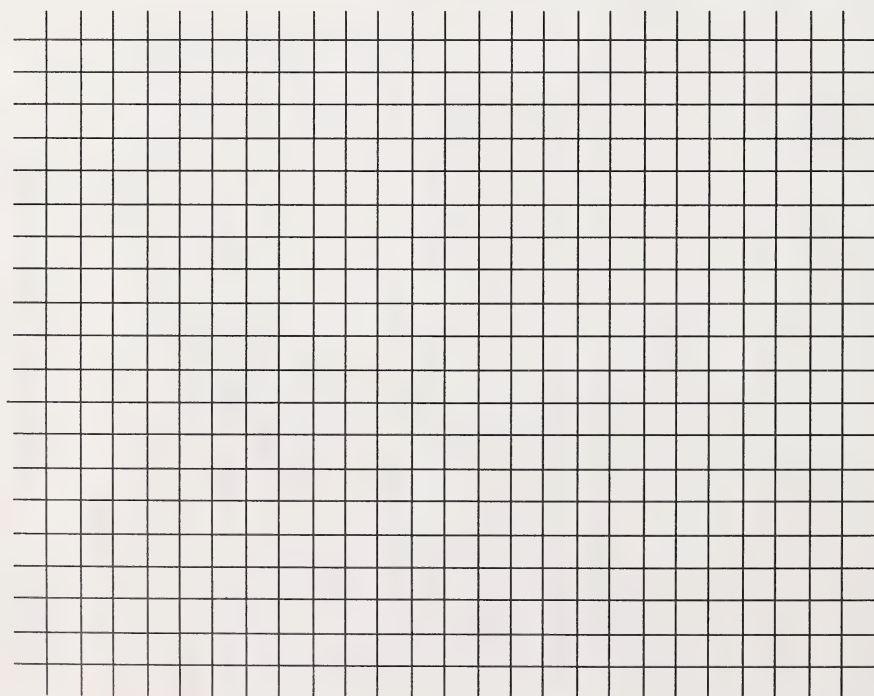
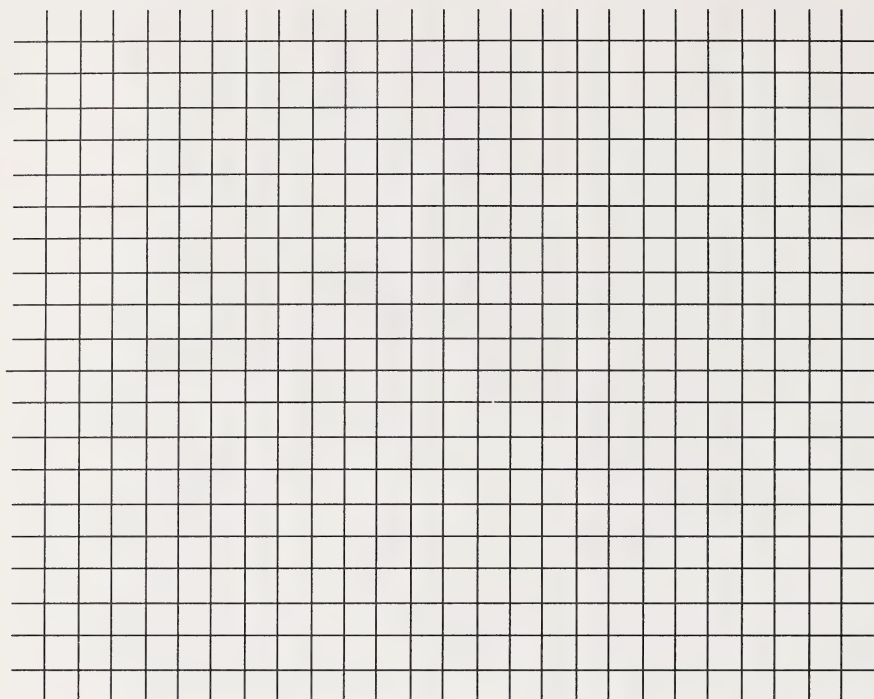


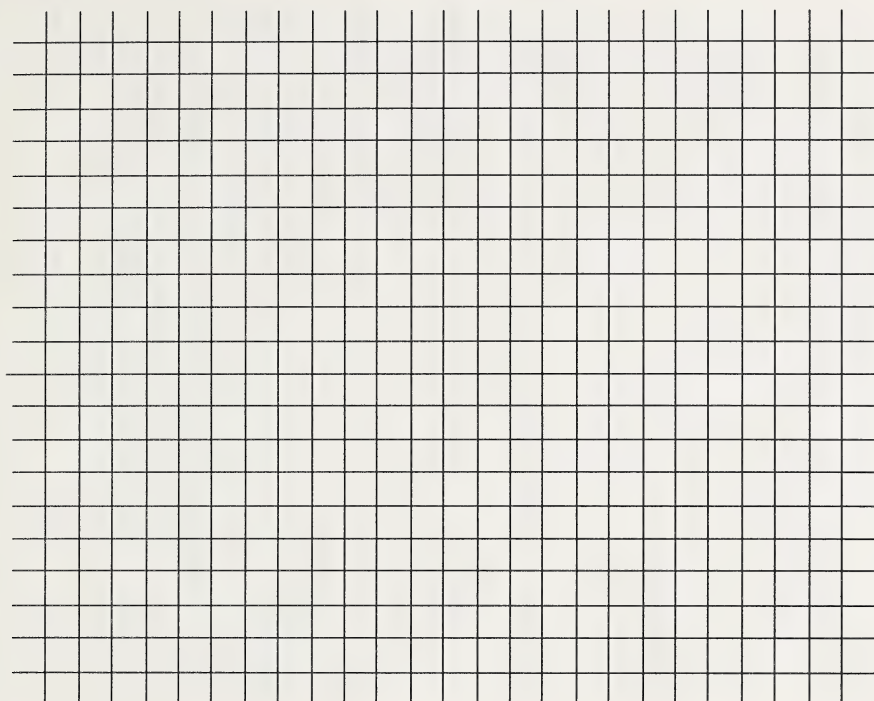
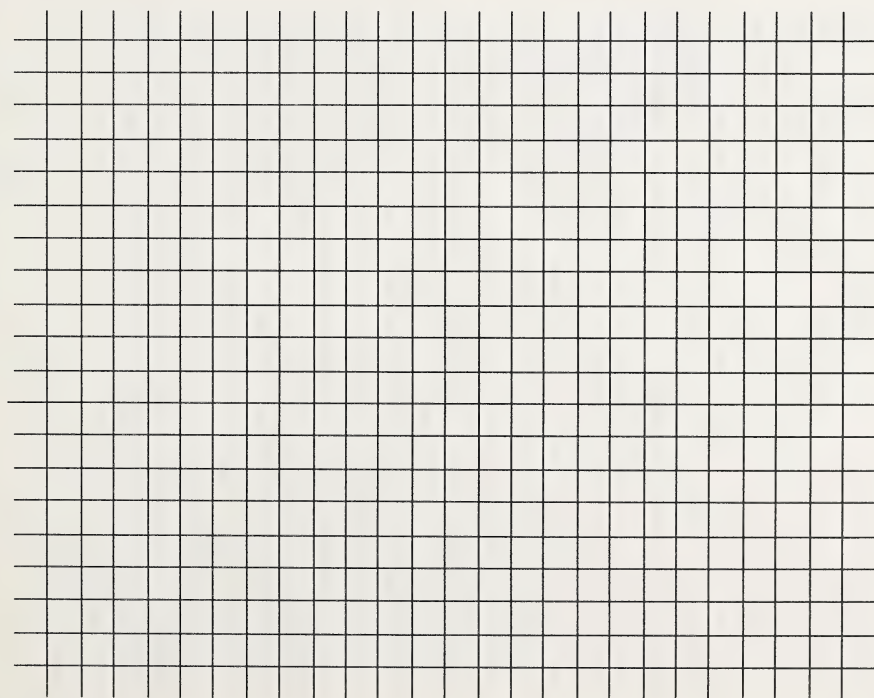
Appendix B Graphing Material

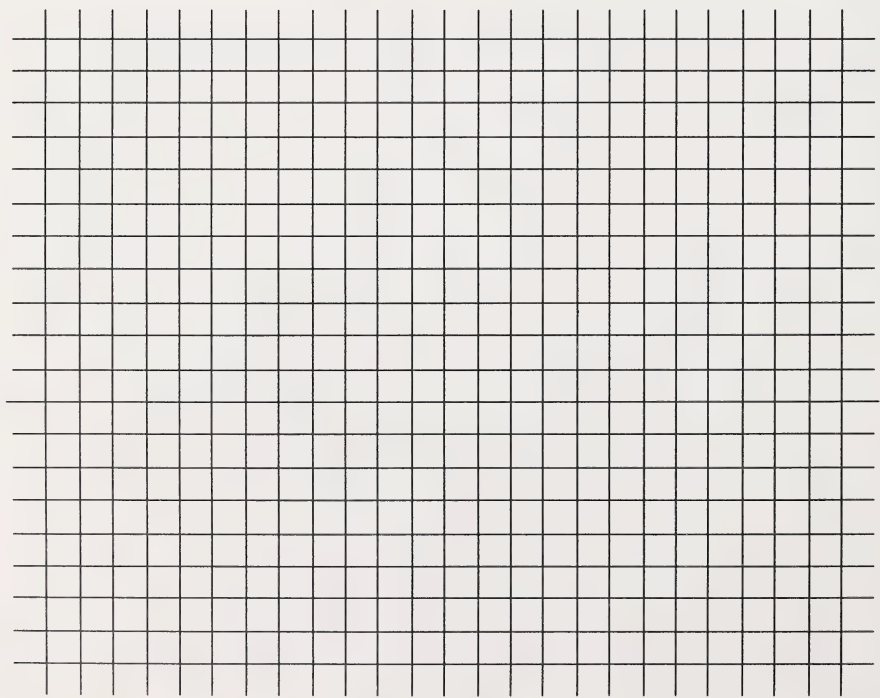
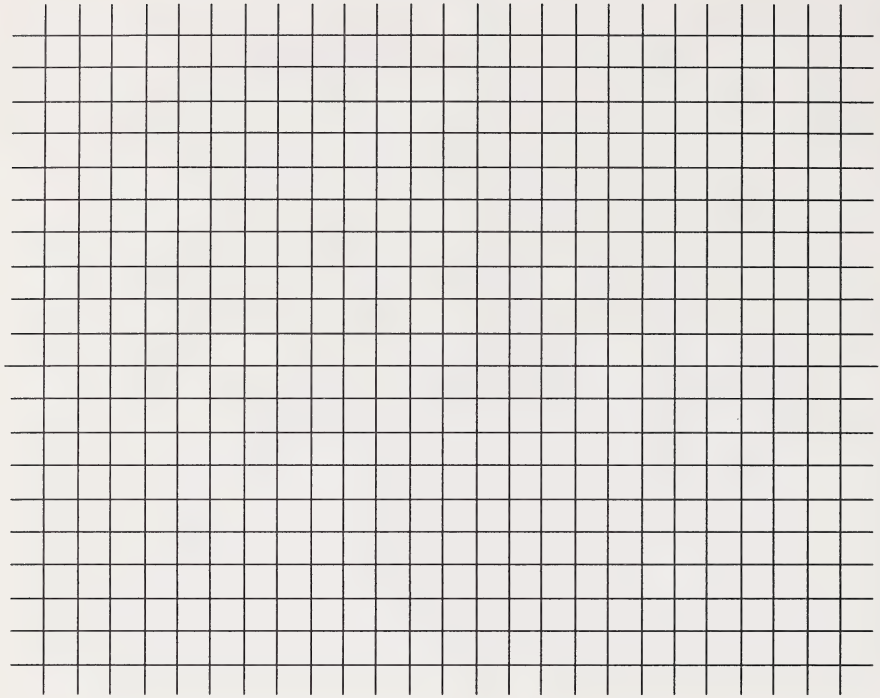
Graph Paper

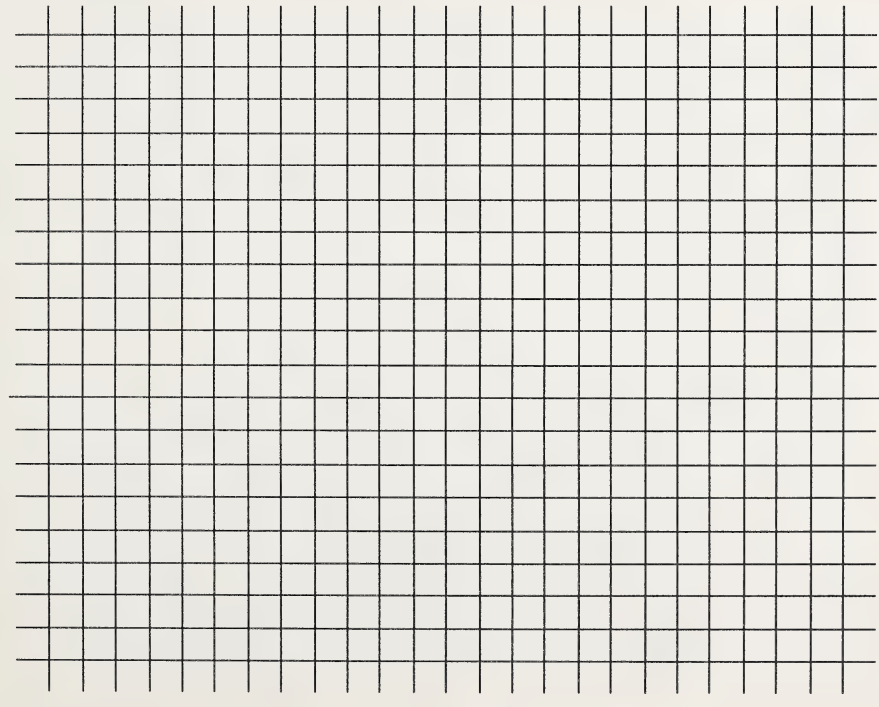
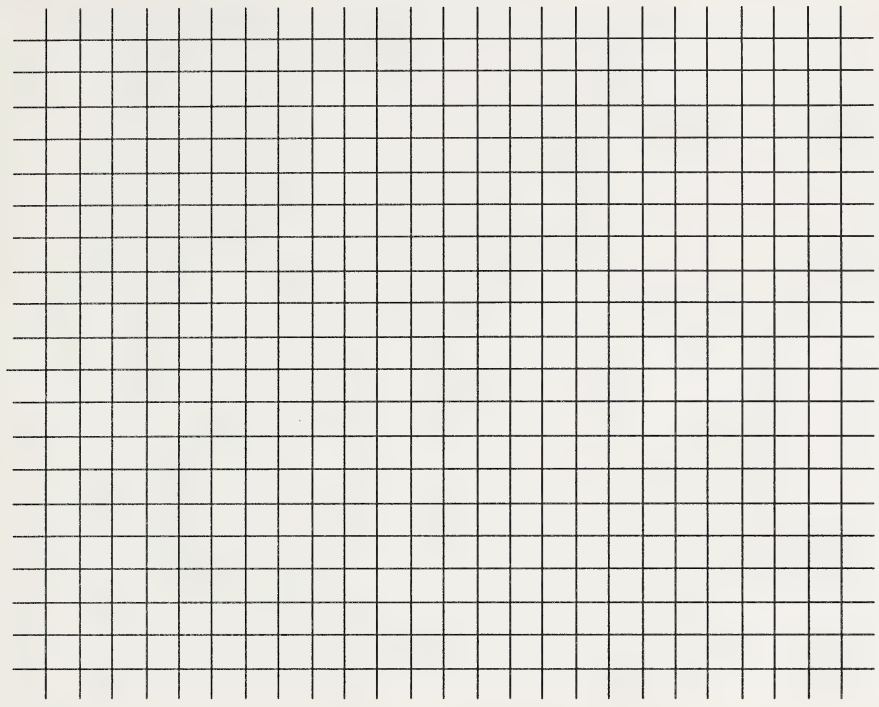


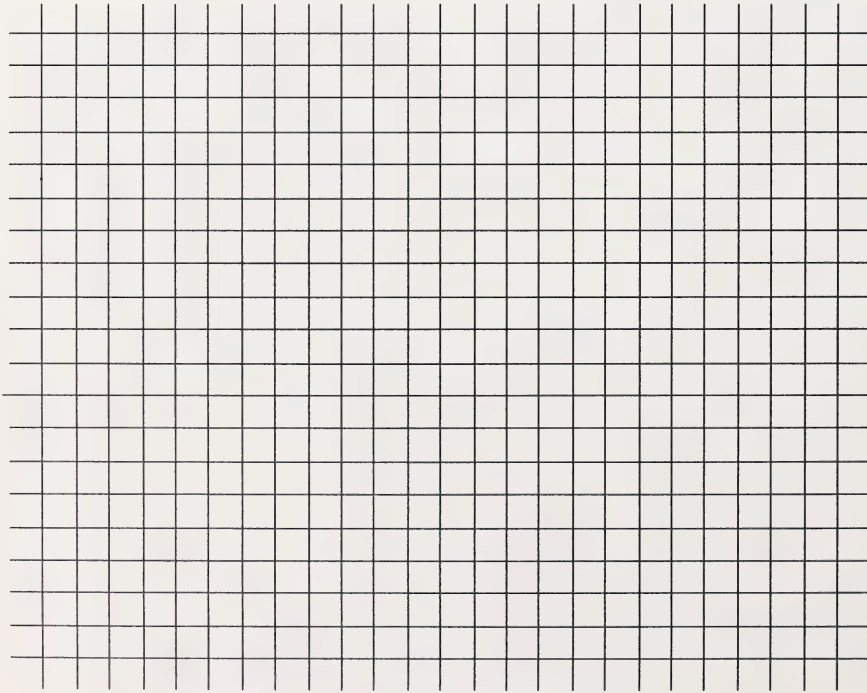
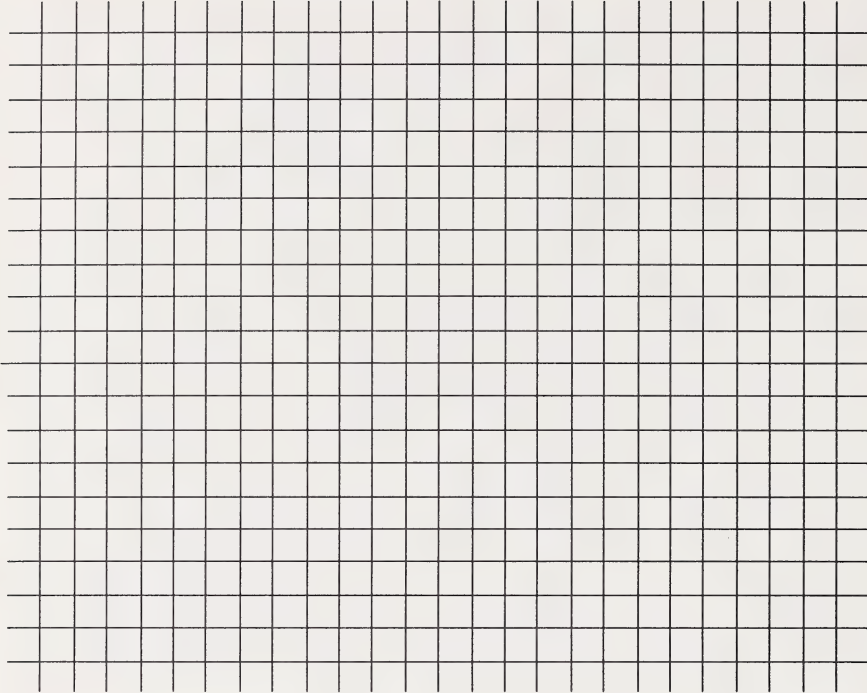


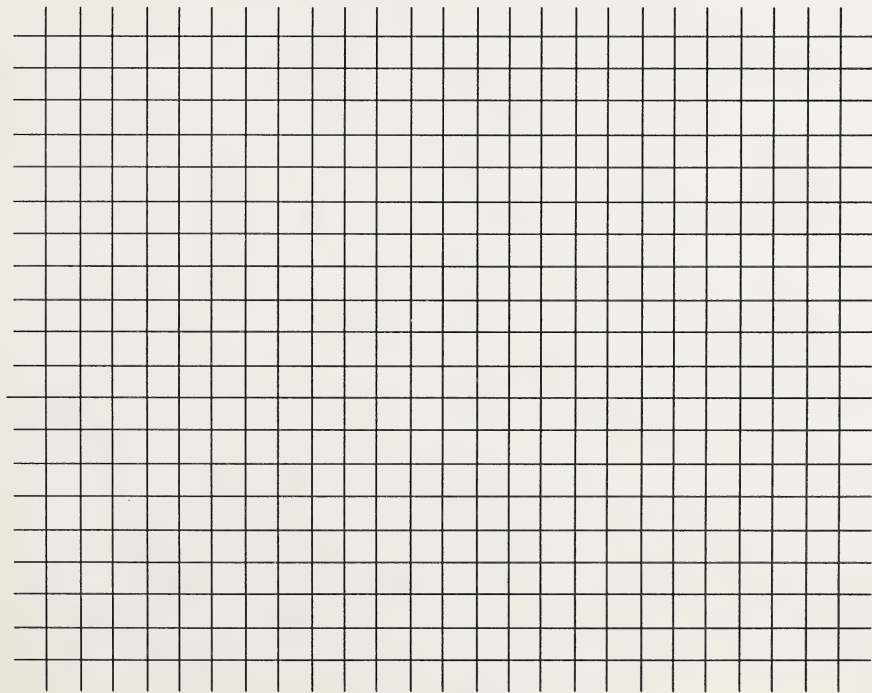
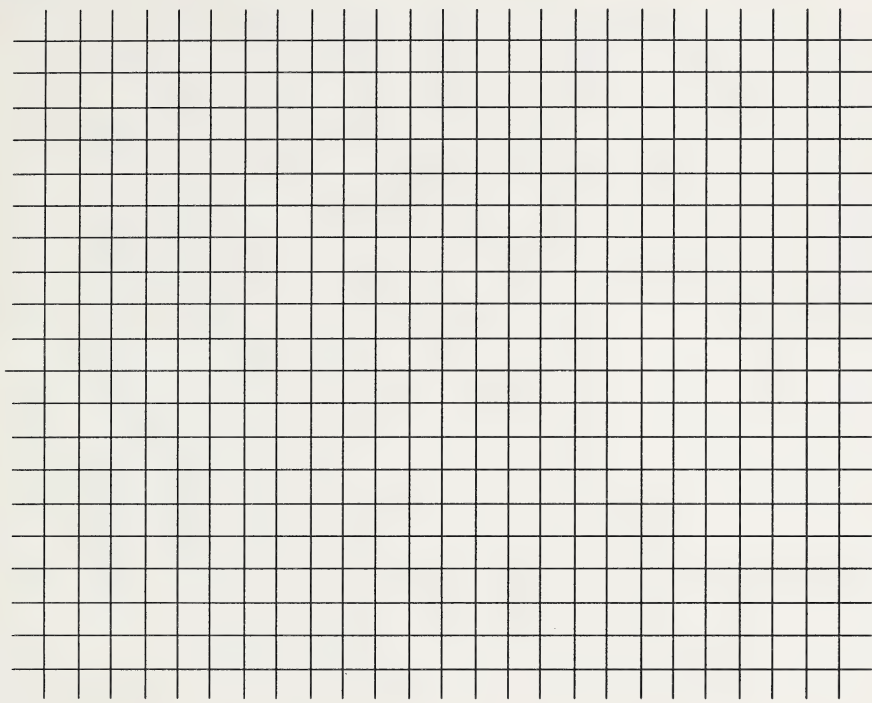


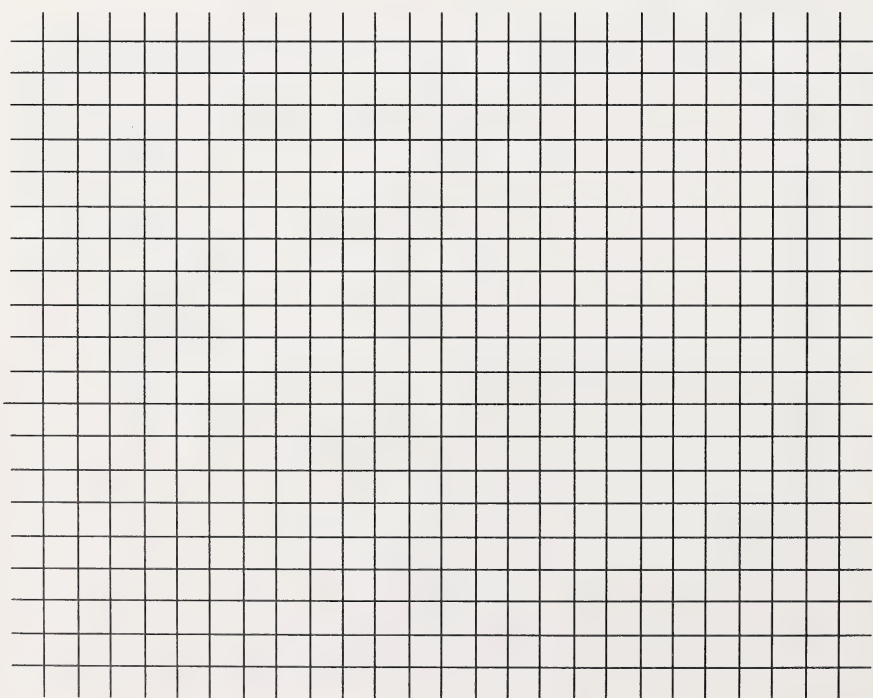
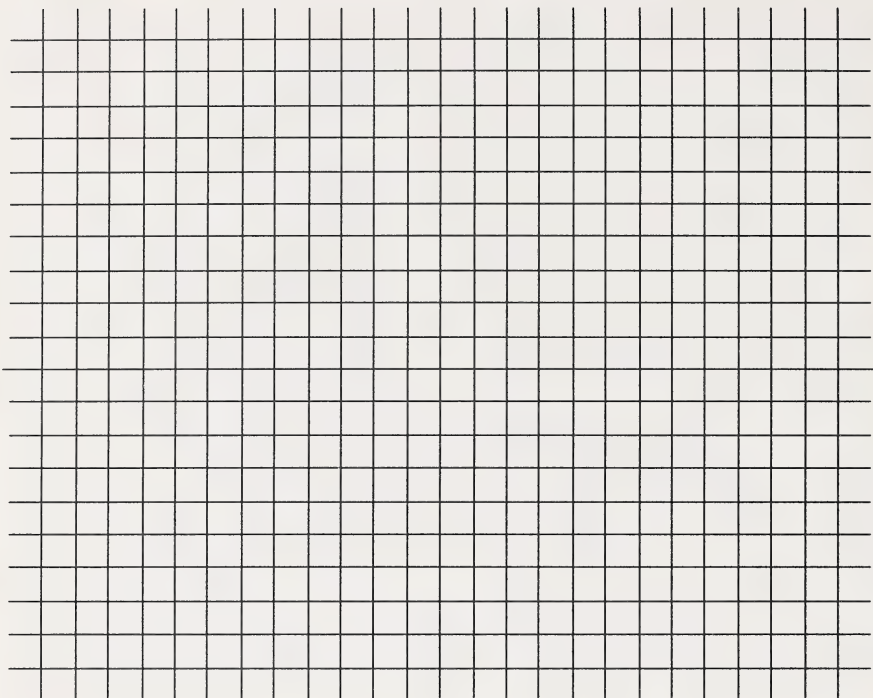


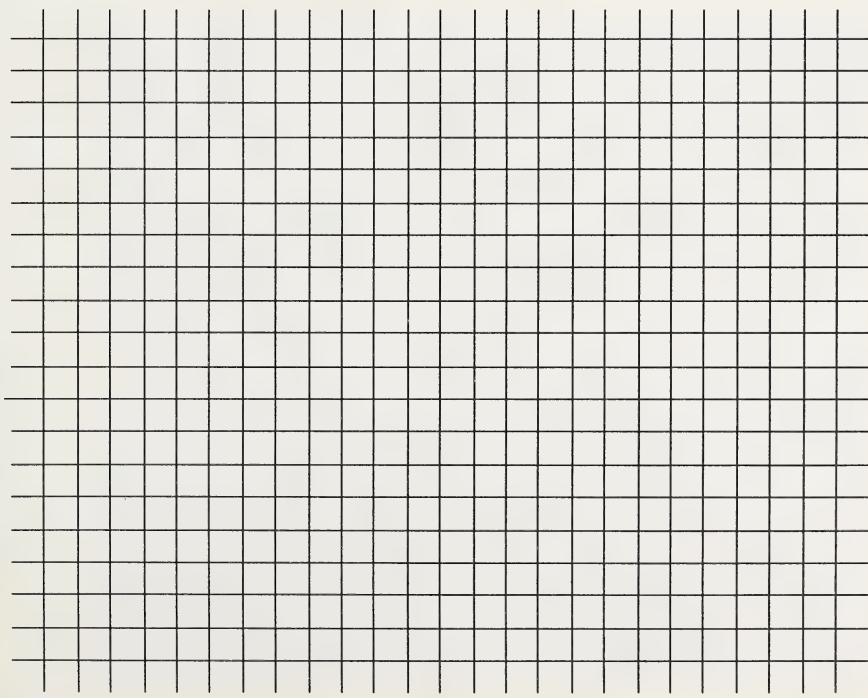
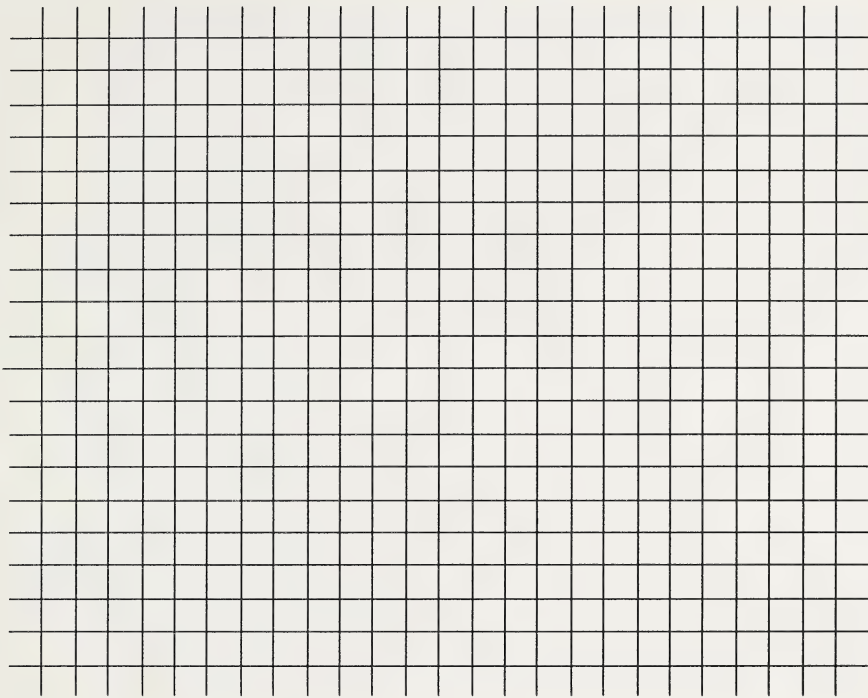


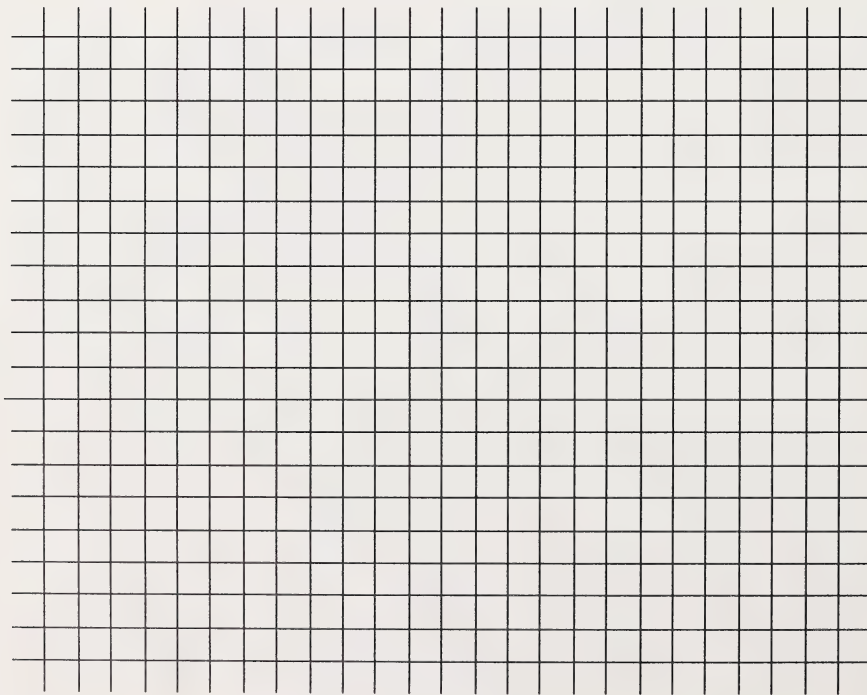
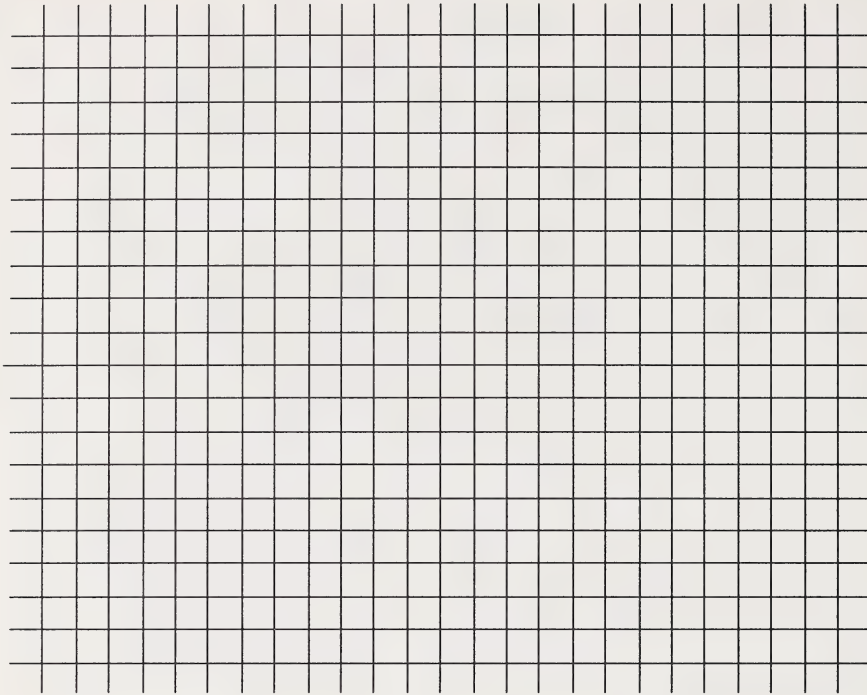


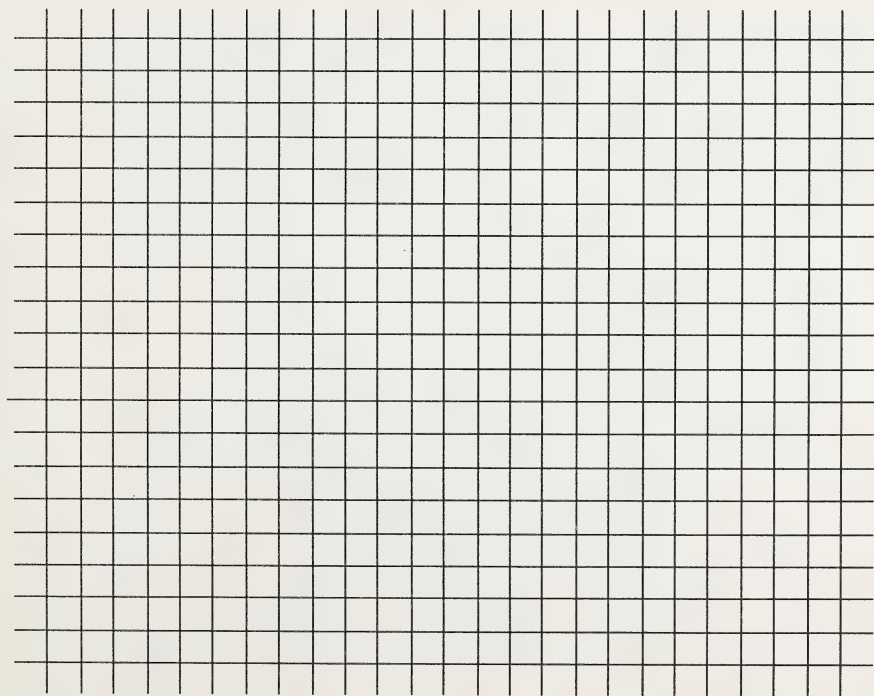
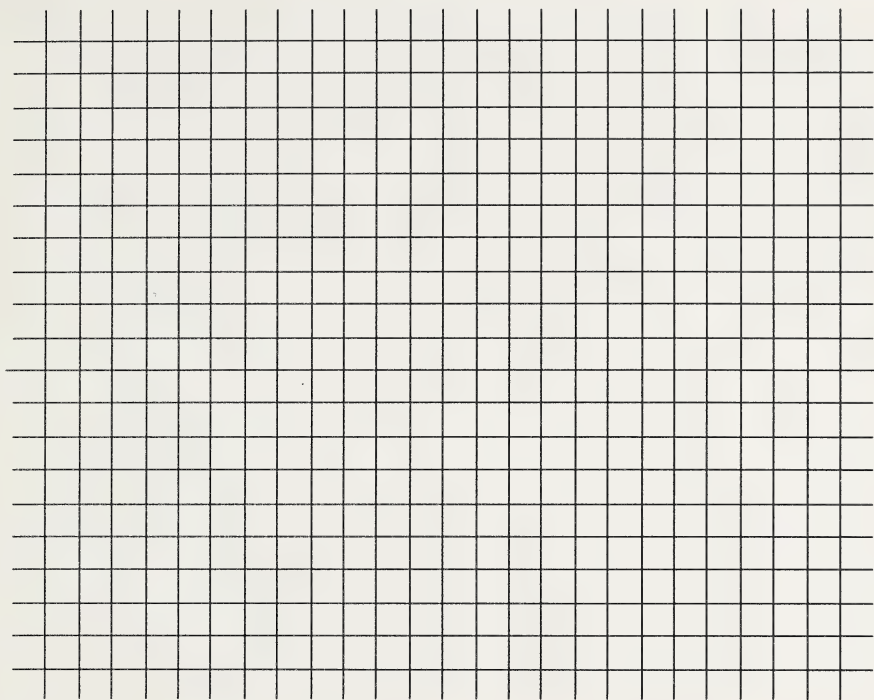


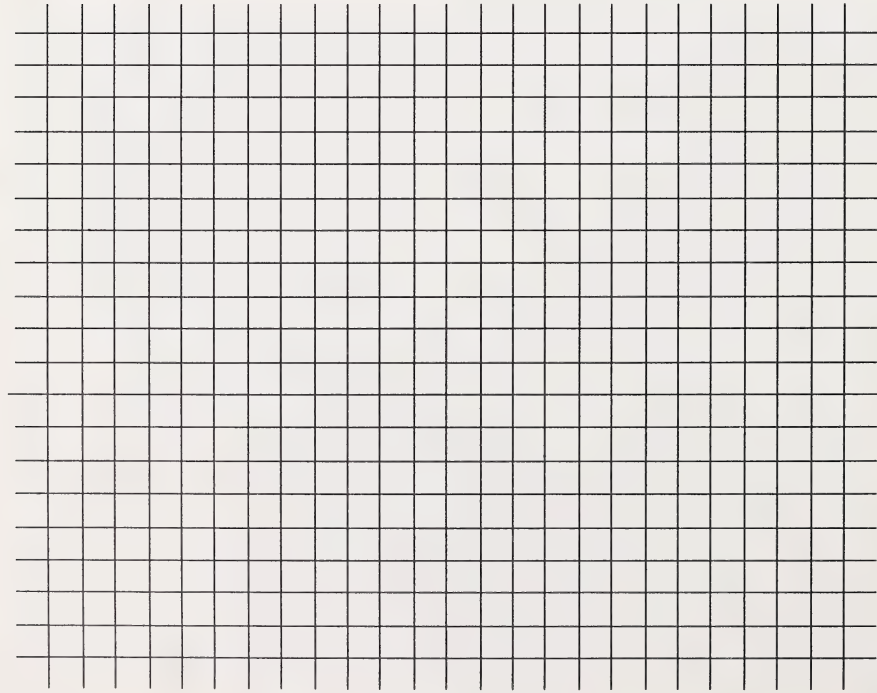
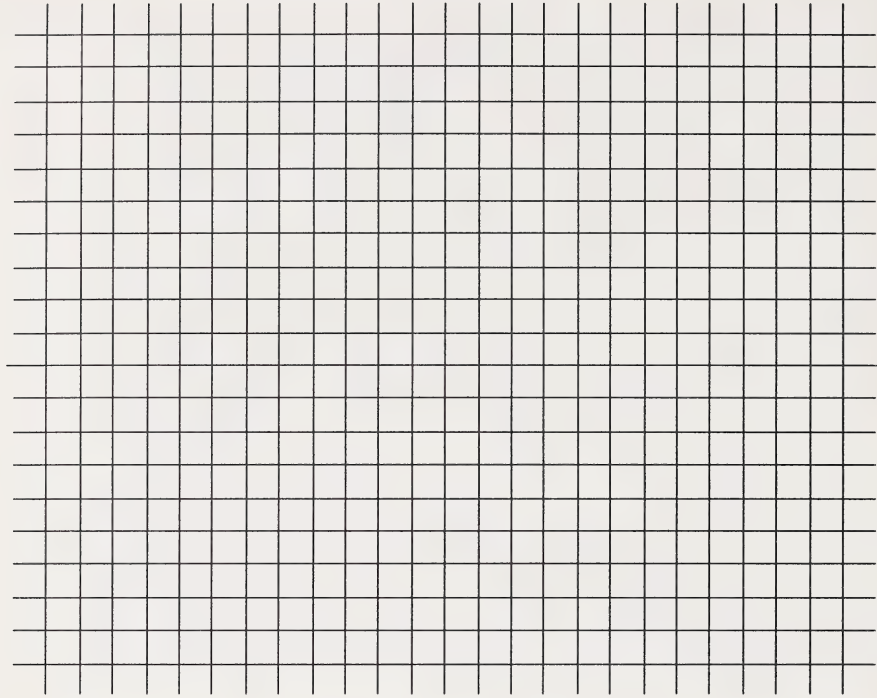


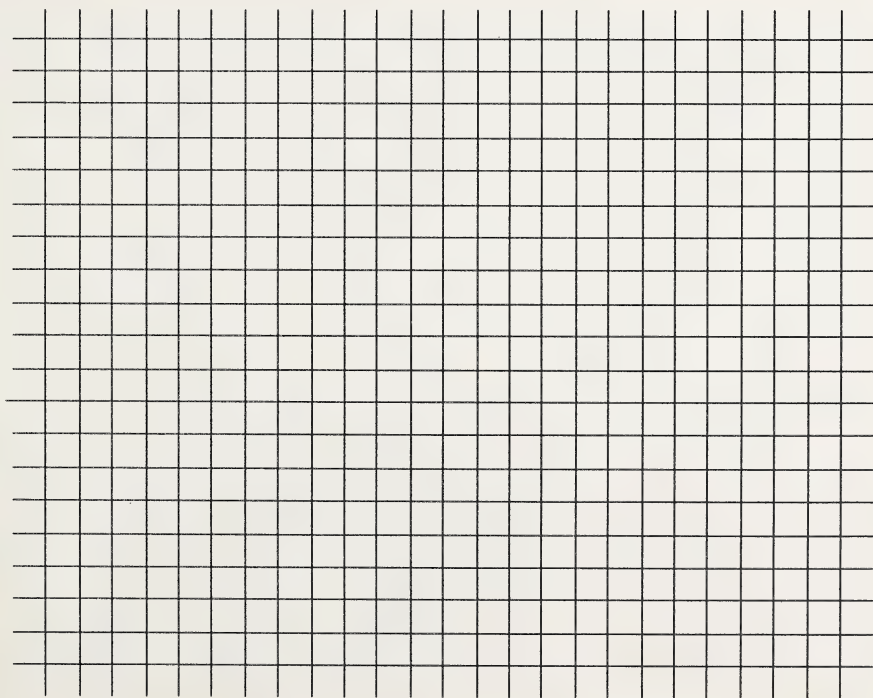
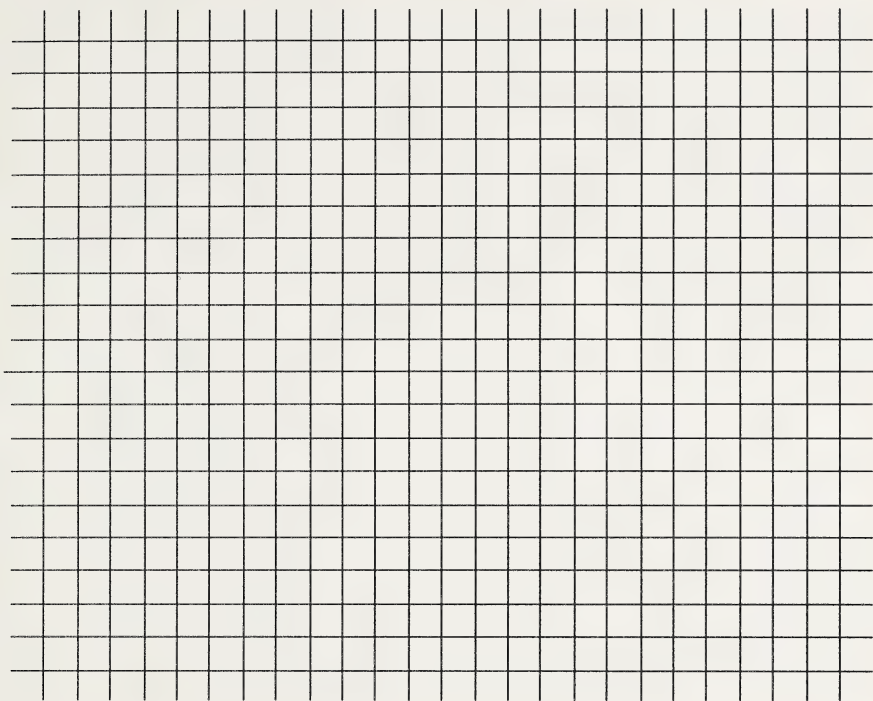


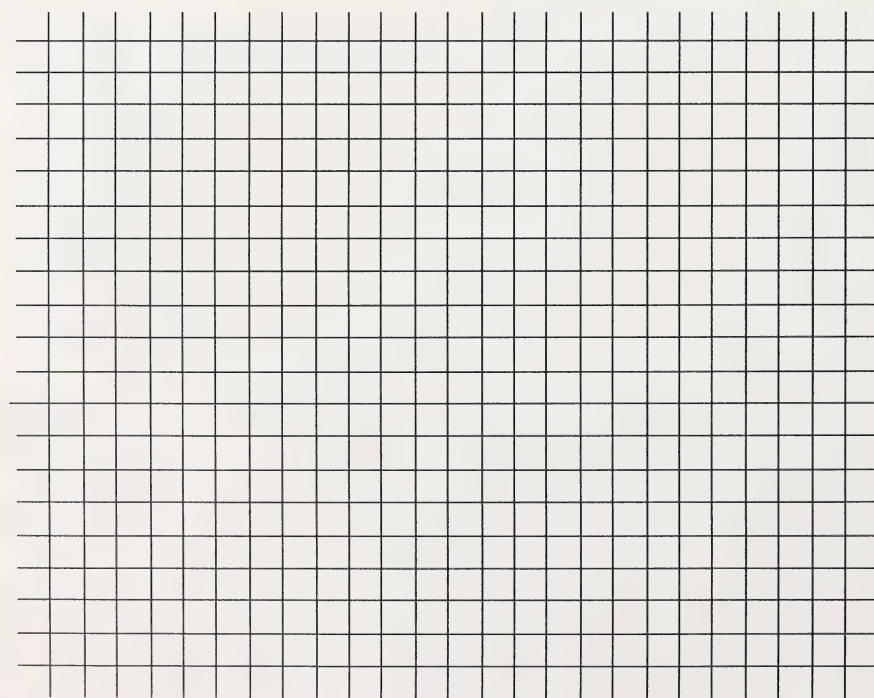
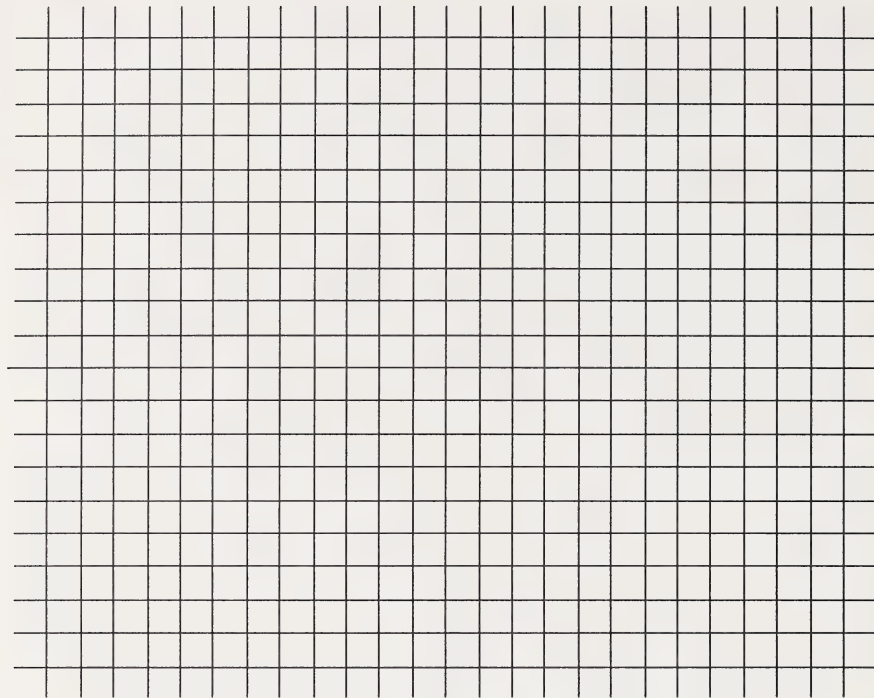


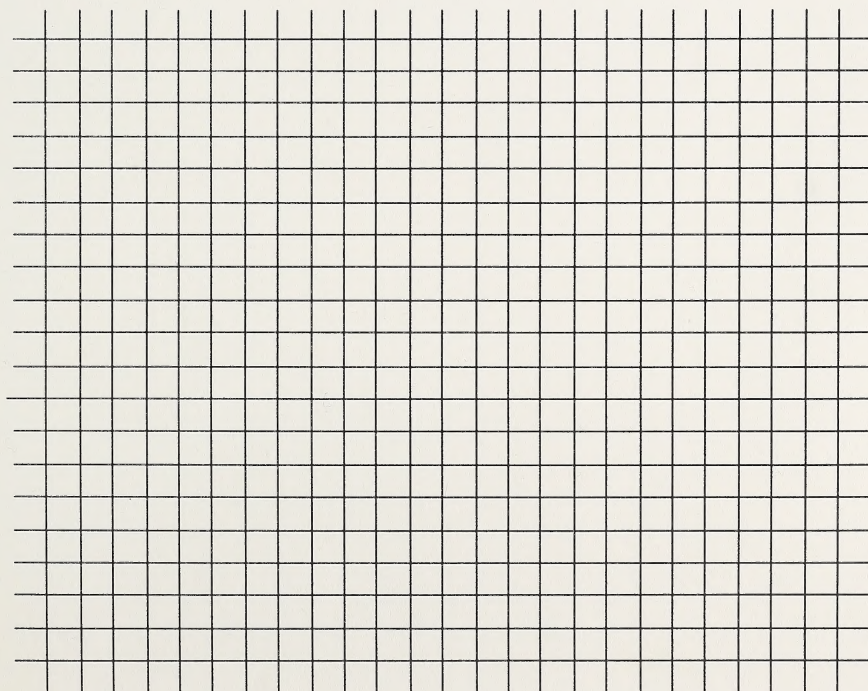
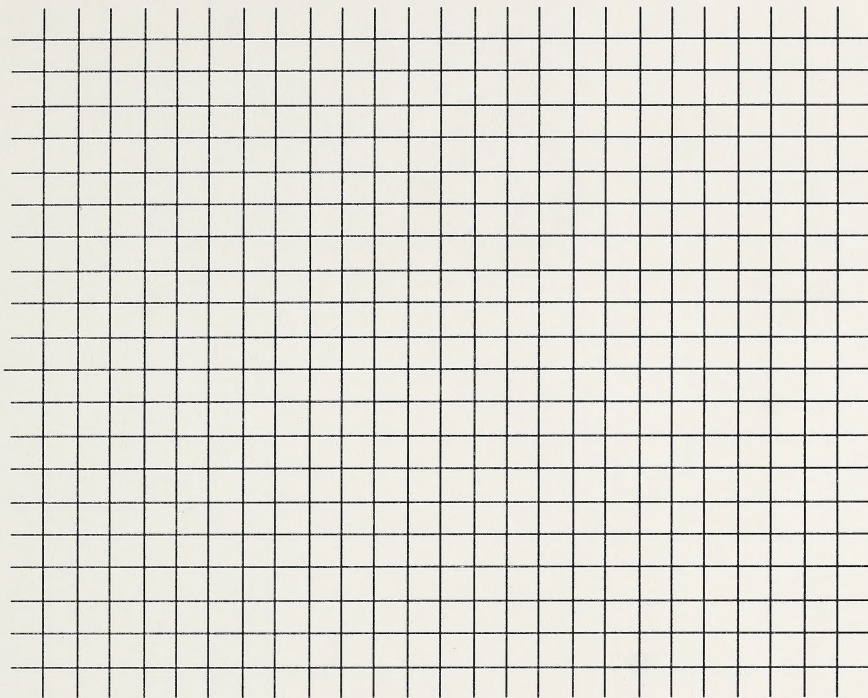












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